

**Mass Generation in QCD - Oscillating Quarks and Gluons  
based in part on 'Tuning to harmonic numbers of oscimodes of  
baryons'**

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**Abstract**

The present lecture is devoted to embedding the approximate genuine harmonic oscillator structure of valence  $q \bar{q}$  mesons and in more detail the  $q q q$  configurations for u,d,s flavored baryons in QCD for three light flavors of quark. It includes notes, preparing the counting of 'oscillatory modes of  $N_{fl} = 3$  light quarks, u , d , s in baryons', using the  $SU(2N_{fl} = 6) \times SO3(\vec{L})$  broken symmetry classification, extended to the harmonic oscillator symmetry of 3 paired oscillator modes.  $\vec{L} = \sum_{n=1}^{N_{fl}} \vec{L}_n$  stands for the space rotation group generated by the sum of the 3 individual angular momenta of quarks in their c.m. system. The oscillator extension to valence gauge boson states is not yet developed to a comparable level .

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## 1 - Introduction

The two main mass generation mechanisms within a general gauge field theory – in  $3 + 1$  uncurved space-time dimensions – henceforth called gravitationless gauge field theory –

– minimally the neutrino mass extended standard model based on the gauge group  $SU3_c \times SU2_L \times U1_y$  and one scalar doublet with respect to  $SU2_L$  – form the basis of the present outline.

- 1) the Bose condensation of some components of elementary scalar fields  
scalar stands here for scalar and/or pseudoscalar Yukawa couplings
- 2) the Bose condensation of the gauge field-strength bilinear  
– gauge- and renormilzation group invariant with respect to the *unbroken* gauge group  $SU3_c$  –

Table 1

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The key features of point 1) in table 1) are outlined in section 2.

The main topic here, point 2) in table 1, is elaborated on in section 3, and worked out in more detail in sections 4 - 6 and references quoted therein .

## 2 - Bose condensation of elementary scalar fields ; the Brout-Englert-Higgs effect

Recent assessments can be found in the talks of François Englert – [1-2013] and, more historically oriented, Peter Higgs [2-2013] at the Nobel Prize 2013 awards ceremony.

We base the general properties of spontaneous gauge breaking of an enveloping gravitationless gauge field theory , based on a gauge group  $G_{env} \supset G_{min}$  in the sense of gauge group unification beyond the minimal case  $G_{min} = SU3_c \times SU2_L \times U1_y$  , presented in the introduction.

A minimal such enveloping gauge group is

$$G_{env}^{min} = SO(10) \equiv spin(10) \quad (1)$$

as discussed in refs. [3-1975] and [4-2008] .

Within larger *simple* enveloping groups the exceptional chain

$$G_{env} \rightarrow E6 \subset E8 \supset E6 \times SU(3) \quad (2)$$

is singled out [5-1976], [6-1980] , most importantly because it offers the possibility of canceling all gauge- and gravitational anomalies in the product gauge group [7-1984] , [8-2012]

$$G_{env} = E8 \times E8 \quad (3)$$

Following the hypothesis of an underlying unifying gauge group the top down of gauge breaking is initiated by a primary breaking followed last by the electroweak gauge breaking .

$$M_{env} \gg v = \left( \sqrt{\sqrt{2} G_F} \right)^{-1} = 246.220 \text{ GeV} \quad (4)$$

In eq. 4  $v$  denotes the v.e.v. of the unique doublet scalar field using the quaternion associated basis for the local scalar fields

$$\begin{aligned} z(x) &= \begin{pmatrix} \varphi^0 & -(\varphi^-)^* \\ \varphi^- & (\varphi^0)^* \end{pmatrix} (x) \\ \varphi^0 &= 1 / \sqrt{2} (Z_0 - i Z_3) ; \varphi^- = 1 / \sqrt{2} (Z_2 - i Z_1) \\ Z_\mu &= (Z_\mu)^* = (Z_0, Z_1, Z_2, Z_3) = (Z_0, \vec{Z}) \\ \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_\mu &= (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\sigma_0, \vec{\sigma}) \end{aligned} \quad (5)$$

In eq. 5 the symbol  $*$  denotes hermitian conjugation of individual complex and/or real field components.

Thus the quantity  $z$ , defined in eq. 5, shows its quaternionic representation

$$z(x) = \frac{1}{\sqrt{2}} \left( Z_0 \sigma_0 + \sum_{k=1}^3 Z_k \frac{1}{i} \sigma_k \right) (x) \quad (6)$$

The four  $2 \times 2$  matrices, displayed in eq. 5, form a 1 to 1 true representation of the base quaternions

$$\begin{aligned} \sigma_0 &\leftrightarrow q_0 = \mathbb{1} ; \frac{1}{i} \sigma_m \leftrightarrow q_m \text{ for } m = 1, 2, 3 \\ q_m q_n &= -\delta_{mn} q_0 + \varepsilon_{mnr} q_r \text{ for } m, n, r = 1, 2, 3 \end{aligned} \quad (7)$$

As we will see, the final stage of the ( nu-mass extended - ) standard model gauge breaking involving *just* 1 doublet of scalars with respect to the electroweak part  $SU(2)_w \times \mathcal{Y}_w$  represents a case for perfectly semiclassical, driven Bose condensation, eventually contrasting with intrinsic properties of primary breakdown.

This is so, because there is precisely 1 invariant

$$I(z, z^\dagger) = z z^\dagger = \frac{1}{2} \sum_{\mu=0}^3 (Z_\mu)^2 \quad (8)$$

with respect to  $SU(2)_w \times \mathcal{Y}_w$  of which a general invariant is a function. This would not remain true for more than one scalar doublet.

As a consequence of eqs. 4 - 8 , the electroweak gauge breaking is *driven and semiclassical*

$$\langle \Omega | z(x) | \Omega \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \text{ with } v = 246.220 \text{ GeV} \quad (9)$$

independent of  $x$

We shall discuss 2 types of primary gauge breaking denoted a) and b) below. The v.e.v.  $v$  in eq. 9 corresponds to the classical minimum of the quartic potential , uniquely restricted to depend on two parameters

$$V(z, z^\dagger) = \left[ -\mu^2 z z^\dagger + \lambda (z z^\dagger)^2 \right]_{11} \quad (10)$$

The minimum conditions become

$$\begin{aligned} \partial_{Z_\nu} V &= \left( -\mu^2 + \lambda |Z|^2 \right) Z_\nu = 0 \longrightarrow z z^\dagger|_{11} = \frac{1}{2} \mu^2 / \lambda \\ V|_{min} &= -\frac{1}{4} (\mu^4 / \lambda) \end{aligned} \quad (11)$$

The second derivatives with respect to  $Z$  at the minimum of the potential become

$$\begin{aligned} \frac{1}{2} \partial_{Z_\rho} \partial_{Z_\sigma} V &= \lambda Z_\rho Z_\sigma |_{min} = \lambda v^2 \delta_{0\rho} \delta_{0\sigma} \\ Z_\nu |_{min} &= (v, \vec{0}) \end{aligned} \quad (12)$$

Expanding the deviation of the potential up to quadratic terms around the minimum thus yields

$$(\Delta V)^{(2)} = \lambda v^2 (\Delta Z_0)^2 ; Z_0(x) = v + \Delta Z_0(x) \quad (13)$$

It is customary to denote the shifted hermitian field  $\Delta Z_0(x)$

$$Z_0 = v + \Delta Z_0 ; \Delta Z_0(x) = H(x) \quad (14)$$

From eq. 14 we read off the mass of the field  $H(x)$  as well as the vanishing of the mass of the other three fields

$$\begin{aligned} m_H^2 &= 2\lambda v^2 = 2\mu^2 ; m_{\vec{Z}}|_{from V} = 0 \\ \vec{Z}(x) &= (Z_1, Z_2, Z_3)(x) \end{aligned} \quad (15)$$

However in the case at hand the existence of would-be long range forces represented by the  $SU(2)_L \times U(1)_Y$  gauge field interactions does not permit the existence of goldstone-modes .

The three would-be Goldstone fields, defined in eq. 15, through their space-time gradient, mix with the gauge bosons to become massive, vis.  $W^\pm$  and  $\tilde{Z}$

$$\partial_{x^\tau} Z_m(x) \leftrightarrow W_\tau^\pm(x), \tilde{Z}_\tau(x) \quad (16)$$

in such a way as to obtain masses of the 3 massive gauge bosons  $m_W, m_{\tilde{Z}}$  and physically form the longitudinal spin components of the resulting massive states (resonances) . In eq. 16 the neutral massive gauge boson is denoted  $\tilde{Z}$  not to confuse it with the scalar field components  $Z_0, \vec{Z}$  . In tree approximation the mass-square of the H-field is twice the value of the  $Z_0$  field in the unbroken case, i.e. for  $\mu^2 \rightarrow -\mu^2$

$$m_H^2 = 2\mu^2 \rightarrow \mu = \frac{1}{\sqrt{2}} m_H = 88.388 \text{ GeV for } m_H = 125 \text{ GeV} \quad (17)$$

The detailed description of the mixing as stated in eq. 16 is not given here. A complete derivation can be found in the textbook [9-1995] .

The simplicity of the presumably lowest in scale gauge breaking relative to the SM gauge group  $SU(2)_L \times U(1)_Y$  prompted most discussions of primary gauge breaking to be of the same type b) presented below, i.e. driven - semiclassical: for an example of primary gauge breakdown patterns see e.g. ref. [10-1013] .

This brings us to the two types a) and b) – of an enveloping gravitationless gauge field theory. For both types the scalar field variables turn out to involve a *complex* ensemble of irreducible representations of the enveloping gauge group – as e.g.  $SO(10)$  , in particular for the generation of masses for neutrino flavors, both light and heavy, as discussed e.g. in refs. 4 – [4-2008] and 10 – [10-1013] through Yukawa couplings to basic fermion bilinears . To this end we introduce notation for the ensemble of scalar fields adapted to primary gauge breaking, generalizing  $SU(2)_L$  doublets as defined in eqs. 5 - 7 .

Lets fix for definiteness

$$G_{env}^{min} = SO(10) \equiv spin(10) \quad (18)$$

in the following. A general irreducible representation of  $SO(10)$  shall be denoted  $[\mathcal{D}]$  , where  $\mathcal{D}$  is equivalenced to its dimension. As entry point we take the [16] representation for one fermion family ( the lightest in mass ) in the left chiral basis

$$[16] : \left( \begin{array}{cccc|cccc} \textcolor{red}{u}^1 & \textcolor{green}{u}^2 & \textcolor{blue}{u}^3 & \nu_e & \mathcal{N}_e & \hat{u}^3 & \hat{u}^2 & \hat{u}^1 \\ \textcolor{red}{d}^1 & \textcolor{green}{d}^2 & \textcolor{blue}{d}^3 & e^- & e^+ & \hat{d}^3 & \hat{d}^2 & \hat{d}^1 \end{array} \right)^{\hat{\gamma} \rightarrow L} \quad (19)$$

$$= (f)^{\hat{\gamma}}$$

## 2-1 - Primary gauge breaking

Following the hypothesis of an underlying unifying gauge group the top down of gauge breaking is initiated by a primary breaking followed last by the electroweak gauge breaking .

Primary gauge beaking is linked to the unifying gauge group scale  $M_{env}$  assumed and also restricted by limits on direct observation of baryon decays

and lepton flavor violation to be much larger than the electroweak scale, defined in eq. 4, repeated below

$$M_{env} \gg v = \left( \sqrt{\sqrt{2} G_F} \right)^{-1} = 246.220 \text{ GeV} \quad (20)$$

In eq. 20  $v$  denotes the v.e.v. of the unique doublet scalar field using the quaternion associated basis for the local scalar fields, defined in eq. 5, where the symbol  $*$  denotes hermitian conjugation of individual complex and/or real field components.

Thus the quantity  $z$ , defined in eq. 5, shows its quaternionic representation, as defined in eq. 6 repeated below

$$z(x) = \frac{1}{\sqrt{2}} \left( Z_0 \sigma_0 + \sum_{k=1}^3 Z_k \frac{1}{i} \sigma_k \right) (x) \quad (21)$$

The four  $2 \times 2$  matrices, displayed in eq. 5, form a 1 to 1 true representation of the base quaternions as given in eq. 7, repeated below

$$\begin{aligned} \sigma_0 &\leftrightarrow q_0 = \mathbb{1} ; \quad \frac{1}{i} \sigma_m \leftrightarrow q_m \text{ for } m = 1, 2, 3 \\ q_m q_n &= -\delta_{mn} q_0 + \varepsilon_{mnr} q_r \text{ for } m, n, r = 1, 2, 3 \end{aligned} \quad (22)$$

As we will see, the final stage of the (nu-mass extended -) standard model gauge breaking involving *just* 1 doublet of scalars with respect to the electroweak part  $SU(2)_w \times \mathcal{Y}_w$  represents a case for perfectly semiclassical, driven Bose condensation, eventually contrasting with intrinsic properties of primary breakdown.

This is so, because there is precisely 1 invariant

$$I(z, z^\dagger) = z z^\dagger = \frac{1}{2} \sum_{\mu=0}^3 (Z_\mu)^2 \quad (23)$$

with respect to  $SU(2)_w \times \mathcal{Y}_w$  of which a general invariant is a function. This would not remain true for more than one scalar doublet.

As a consequence of eqs. 4 - 8, the electroweak gauge breaking is *driven and semiclassical*

$$\langle \Omega | z(x) | \Omega \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \text{ with } v = 246.220 \text{ GeV} \quad (24)$$

independent of  $x$

We shall discuss 2 types of primary gauge breaking denoted a) and b) below.

The simplicity of the presumably lowest in scale gauge breaking relative to the SM gauge group  $SU(2)_L \times U(1)_Y$  prompted most discussions of primary gauge breaking to be of the same type b) presented below, i.e. driven - semiclassical: for an example of primary gauge breakdown patterns see e.g. ref. [10-1013].

This brings us to the two types a) and b) – of an enveloping gravitationless gauge field theory. For both types the scalar field variables turn out to

involve a *complex* ensemble of irreducible representations of the enveloping gauge group – as e.g.  $SO(10)$ , in particular for the generation of masses for neutrino flavors, both light and heavy, as discussed e.g. in refs. 4 – [4-2008] and 10 – [10-1013] through Yukawa couplings to basic fermion bilinears . To this end we introduce notation for the ensemble of scalar fields adapted to primary gauge breaking, generalizing  $SU2_L$  doublets as defined in eq. 19, repeated below, fixing for definiteness

$$G_{env}^{min} = SO(10) \equiv spin(10) \quad (25)$$

in the following. A general irreducible representation of  $SO(10)$  shall be denoted  $[\mathcal{D}]$ , where  $\mathcal{D}$  is equivalenced to its dimension. As entry point we take the [16] representation for one fermion family ( the lightest in mass ) in the left chiral basis as defined in eq. 19, repeated below

$$[16] : \left( \begin{array}{cccc|cccc} \textcolor{red}{u}^1 & \textcolor{green}{u}^2 & \textcolor{blue}{u}^3 & \nu_e & \mathcal{N}_e & \hat{u}^3 & \hat{u}^2 & \hat{u}^1 \\ \textcolor{red}{d}^1 & \textcolor{green}{d}^2 & \textcolor{blue}{d}^3 & e^- & e^+ & \hat{d}^3 & \hat{d}^2 & \hat{d}^1 \end{array} \right)^{\dot{\gamma} \rightarrow L} \quad (26)$$

$$= (f)^{\dot{\gamma}}$$

**the Majorana logic characterized by  $\mathcal{N}_{e,\mu,\tau}$**

We illustrate the use of the basis defined in eq. 26 considering the group decomposition

$$spin(10) \rightarrow SU5 \times U1_{J_5} \quad (27)$$

Among the 3 generators of  $spin(10)$  commuting with  $SU3_c$ ,  $I_{3L}$ ,  $I_{3R}$  and Cartan subalgebra of  $spin(10)$  there is one combination, denoted  $J_5$  in eq. 27, commuting with its largest unitary subgroup  $SU5$ . The charges  $Q(J_5)$  form the pattern as in eq. 28

$$[16] : Q(J_5) = \left( \begin{array}{cccc|cccc} \textcolor{red}{1} & \textcolor{green}{1} & \textcolor{blue}{1} & -3 & 5 & 1 & 1 & 1 \\ \textcolor{red}{1} & \textcolor{green}{1} & \textcolor{blue}{1} & -3 & 1 & -3 & -3 & -3 \end{array} \right) \quad (28)$$

$Q(J_\mu)$  with charges as given in eq. 28 represents a *hermitian* generator of the Cartan subalgebra of  $spin(10)$ , unique up to a (real) multiplicative factor, which commutes with the the  $SU5$  subgroup of  $spin(10)$ . The flavors within one family sharing the same Q-charges in fact form irreducible representations of  $SU5$ , which shall be labeled

$$\mathcal{D}(SU5) \rightarrow \{\mathcal{D}(SU5)\}_Q \quad (29)$$

with  $\mathcal{D}(SU5)$  equivalenced with the dimension of the representation. The suffix Q is added in eq. 29, since in the present context  $SU5$  multiplets necessarily occur *embedded* in  $SO(10)$  representations.



Thus the Q values of the [16] representation on the right hand side of eq. 28 translate to

$$[16] = \{1\}_5 + \{10\}_1 + \{\bar{5}\}_{-3} \quad (30)$$

The sequence of Q values :  $Q = 5, 1, -3$  as arranged in the sequence on the right hand side of eq. 30 – decreasing in steps of 4 – is related to the properties of binomial coefficients

$$\binom{10}{n} \text{ for } n = 0, 4, 8 \text{ and } n = 10, 6, 2 \quad (31)$$

This is derived in ref. 4 – [4-2008] – and reproduced in part in Fig. 3 at the end of Appendix 1.

It is instructive to decompose the Q-value pattern in eq. 28 into an  $SU_{2L+R}$  invariant part and the remainder proportional to  $I_{3L+R}$

$$\begin{pmatrix} 1 & 1 & 1 & -3 & | & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & | & 1 & -3 & -3 & -3 \\ 1 & 1 & 1 & -3 & | & 3 & -1 & -1 & -1 \\ 1 & 1 & 1 & -3 & | & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & | & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & | & -2 & -2 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -3 & | & 3 & -1 & -1 & -1 \\ 1 & 1 & 1 & -3 & | & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & | & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & | & -2 & -2 & -2 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & -3 & -3 & -3 \\ 3 & -1 & -1 & -1 \\ 3 & -1 & -1 & -1 \end{pmatrix} \quad (32)$$

From eq. 32 we obtain the identification of the  $spin(10)$  Cartan subalgebra hermitian components (charges)

$$Q = 3(B - L) - 4I_{3R} \quad (33)$$

In eq. 33 B and L denote baryon and lepton number respectively .

Two remarks shall follow , concerning the recognizable key features inherent to spontaneous gauge breaking of an enveloping gravitationless gauge field theory , based on a gauge group  $G_{env} = spin(10)$

#### 1) primary gauge breakdown

must be much different than on the lowest – i.e. electroweak – scale level pertaining to  $G_{SM} = SU_{3c} \times SU_{2L} \times U_{1Y}$  .

This is so because empirically well established candidate symmetries , like baryon and lepton number conservation are broken on the primary level and imply very large scale of unification  $M_{env} = O(10^{16} \text{GeV})$ .

As examples let me quote the upper limits of the  $\mu^+ \rightarrow e^+ \gamma$  and  $\mu^+ \rightarrow e^+ + e^+ e^-$  branching fractions

$$\begin{aligned} Br(\mu^+ \rightarrow e^+ + \gamma) &< 2.4 \cdot 10^{-12} \\ Br(\mu^+ \rightarrow e^+ + e^+ e^-) &< 1.0 \cdot 10^{-12} \text{ in ref. 11 – [11-2013]} \end{aligned} \quad (34)$$

2) the power of the set of scalar fields

involved in primary gauge breaking does not follow any principle of minimal selection of *spin* (10) representations pertaining to scalar fields .

### specific notation for scalar field variables

We proceed defining notation for scalar field variables suitable for primary gauge breaking

$$\{\underline{z}\} = \underline{z} \left[ \begin{array}{c} ( \mathcal{D}^{(1)} , \otimes n_1 ) \\ ( \mathcal{D}^{(2)} , \otimes n_2 ) \\ ( \cdots , \otimes \cdots ) \end{array} \right] \text{ |broken down to real coordinates} \quad (35)$$

In eq. 35  $\mathcal{D}^{(\nu)}$  ;  $\nu = 1, 2, \dots$  denote a complete set of unitary irreducible representations of *spin* (10) , finite dimensional; constructively defined through the method of Peter and Weyl [12-1927] .

$n_\nu$  stands for the multiplicity of a given representation  $\mathcal{D}^{(\nu)}$  .

For  $\mathcal{D}^{(\nu)}$  ,  $\overline{\mathcal{D}}^{(\nu)}$  beeing a pair of inequivalent , relative complex conjugate representations the coordinates  $\underline{z}(\mathcal{D}^{(\nu)})$  ,  $\underline{\bar{z}}(\overline{\mathcal{D}}^{(\nu)})$  broken down to real and imaginary parts count as  $2(\text{complex}) \dim(\mathcal{D})$  components over real numbers. This is the meaning of the attribute 'broken down to real coordinates' on the right hand side of eq. 35 .

Thus choosing real values for the components of  $\underline{z} \in R^M$  as defined in eq. 35 it follows

$$\underline{z} = ( z_1 , \cdots , z_r , \cdots , z_M ) ; z_r : \text{hermitian fields} \quad (36)$$

we find

$$M = \sum_\nu n_\nu \left\{ \begin{array}{l} \dim(\mathcal{D}^{(\nu)}) \text{ for } \mathcal{D}^{(\nu)} \text{ real} \\ 2 \text{ complex } \dim(\mathcal{D}^{(\nu)}) \text{ for } \mathcal{D}^{(\nu)} \text{ complex} \end{array} \right\} < \infty \quad (37)$$

### to get an idea of the power of the set of scalars

We illustrate the order of M , in eqs. 36 , 37 by the representations of the fermion bilinears from left and right chiral bases, adapting the scalar variables to their definition in eq. 36 , allowing for complex linear combinations

applied to complex representations (from Appendix 1 and ref. 4, [4-2008] ).

$$\mathcal{H}_{fermion\ mass} \longleftarrow$$

$$\left( \bar{z}^{\overline{126}\ F\ G} \right)_{\bar{\xi}} (f_{a\ 16\ F})_{\dot{\gamma}} (f_{b\ 16\ G})^{\dot{\gamma}} C \left( \begin{array}{c|cc} \overline{126} & 16 & 16 \\ \xi & a & b \end{array} \right) + h.c.$$

$$\left( \bar{z}^{\overline{126}\ F\ G} \right)_{\bar{\xi}} : \text{(pseudo-) scalar fields in the } \overline{126} \text{ representation of SO (10)}$$

$$F, G = I, II, III : \text{fermion family indices}$$

(38)

$$\text{In eq. 38 } C \left( \begin{array}{c|cc} \overline{126} & 16 & 16 \\ \xi & a & b \end{array} \right)$$

denotes the *spin* (10) Clebsch-Gordan coefficients projecting the product of two ( fermionic ) 16 representations on irreducible spin (10) representations. We reproduce the 4 product representations of 16 and  $\overline{16}$  representations from ref. 4 ( Appendix E ) – [4-2008] )

	[16]		$\overline{[16]}$
[16]	$s : \begin{array}{c} [10] + \\ [126] \end{array}, a : [120]$		$[1] + [45] + [210]$
$\overline{[16]}$	$\begin{array}{c} [1] + [45] + \\ [210] \end{array}$		$s : \begin{array}{c} [10] + \\ \overline{[126]} \end{array}, a : [120]$

(39)

The real and complex representations in the multiplication table in eq. 39 are denoted  $\mathcal{D}_R$ ,  $\mathcal{D}_C$  respectively

$$\begin{aligned} \mathcal{D}_R &: [10]_R, [120]_R, [1]_R, [45]_R, [210]_R \\ \mathcal{D}_C &: [126]_C, \overline{[126]}_C \end{aligned} \quad (40)$$

Choosing minimally multiplicities 2 for real and 1 for complex representations in eqs. 39 and 40 yields

$$M = 772 + 252 = 1028 \quad (41)$$

We conclude from the example multiplicities leading to eq. 41

$$M \geq O(1000) \quad (42)$$

### 2-1-a - Primary gauge breaking - type a)

The gauge breaking in type a) *gravitationless gauge field theory* is

1) driven

by pre-established quadratic, cubic and quartic scalar field self interactions

2) *not* reducible to semiclassical approximation

for vacuum expected values for scalar variables and their composite operators, necessarily include gauge variant ones in order to qualify for *gauge breaking*

For clarity let me remark that condition 2) above is necessary, since in the case of *exclusively* gauge invariant vev's for composite scalar operators they are part of an alternative case – in conjunction with other gauge invariant composite field variables – of spontaneous mass generation without gauge breaking. This will be discussed within QCD with 3 light flavors – without scalars – in section 3.

### 2-1-b - Primary gauge breaking - type b)

The gauge breaking in type b) *gravitationless gauge field theory* is

1) driven

by pre-established quadratic, cubic and quartic scalar field self interactions like in type a)

2) reducible to semiclassical approximation

for vacuum expected values for scalar variables and their composite operators. This is the usual case discussed in the literature, as e.g. in refs. 10 – [10-1013] – and 13 – [13-2013] , while in the latter reference the main topic is electroweak gauge breaking .

The semiclassical approximation – with respect to vev's of scalar fields and their composite local operators as well as composite operators involving other fields – means

$$\langle \Omega | f ( \underline{z} ) | \Omega \rangle = f ( \langle \Omega | \underline{z} | \Omega \rangle ) \quad (43)$$

### 3 - Mass Generation in QCD with 3 light flavors - oscillating Quarks and Gluons

In this section we turn to the main topic of this lecture : Mass generation in QCD for three light flavors of quark u,d,s – spontaneous and through the trace anomaly , persisting in the chiral limit  $M_{u,d,s} \rightarrow 0$  .

Here the oscillatory modes of valence quark-antiquark states as well as three valence quark baryon

( antibaryon ) states will be discussed and new results on counting these modes presented.

The next subsections are devoted to embed oscillatory modes of quarks in QCD following lectures given by the author in Erice 2013, ref. 14, [14-2013].

### 3-1 - embedding oscillator modes in u,d,s flavored baryons in QCD

In order to put the main topic of this lecture into perspective let me begin citing ref. [15-1980] , my construction of Poincaré invariant oscillatory modes of three valence quarks ( u , d ) restricted to the two nonstrange flavors in nonstrange baryons . A small collection of references to previous work in this direction is given there ( [15-1980] ) . The disappearing of perfectly gauge invariant explicit dependence on color of quark- and gauge boson-fields is by the confined nature of the oscillator wave functions – restricted to the center of mass system relative space coordinates – relegated to an outer factor

$$\varepsilon_{\alpha\beta\gamma} ; c_{1,2,3} = \text{red , green , blue} \quad (44)$$

$$\alpha, \beta, \gamma = 1, 2, 3 : \text{numbering individual quark positions}$$

The color factor in eq. 44 :  $\varepsilon_{\alpha\beta\gamma}$  , antisymmetric in its three color indices , must be gauge invariant with respect to the local  $SU(3)_c$  gauge group and thus reduced from the 3 positions  $\vec{x}_{1,2,3}$  to a common space point through parallel transport  $q q q$  ( triple ) QCD string factors .The detailed form of the QCD string factors is discussed in section 2 – premises , for which I cite my previous two Erice lectures in refs. [17-2011] , [16-2012] in order to maintain consistency of notation .

A sketch of the  $q \bar{q}$  ( bilocal- or double- ) and  $q q q$  ( triple- ) QCD string factors , also called 'bond structures' , is given in figure 1 - I below

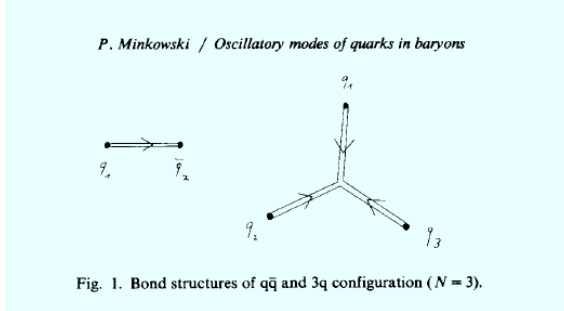


Fig. 1 - I : Bond structures of  $q\bar{q}$  and  $3q$  configurations  
(  $N = 3$  ) from ref. [15-1980]  $\longleftrightarrow$

The next step revived questions related to oscillatory modes of a pair of independent oscillators and their eventual connection to Bogoliubov transformations in November 2010 in ref. [18-2010], an olympic cycle.

Finally on the road leading to the present outline a discussion deserves mention, with the group of Willibald Plessas from the University of Graz during the Oberwölz Symposium 2012 : Quantum Chromodynamics: History and

Prospects 516. WE-Heraeus-Seminar , Oberwölz, Styria, Austria. 3. - 8. September 2012 .

The discussion arose as to whether the baryon modes of light flavors u, d, s of quarks with low total spin were presently more or less completely accounted for – according to the new PDG-review [19-2011] – contrasting with the situation in 1980 with respect to only u and d flavors .

Looking at the exhaustive tables of ref. [19-2011] *included today* the answer is obviously to the negative , yet these tables were not available in the web-version of the PDG tables [20-2012] upon my last search before the Oberwölz Symposium 2012 . Out of this situation the challenge took shape to count the oscillatory modes in baryons in their own right in analytic ways . This work is 'in progres' in collaboration with Sonia Kabana . First results have been reported in ref. 28 – [28-2013] .

With the exception of subsection 3-1-1, quark masses are denoted by capital letters  $M_\alpha$  as at the beginning of section 3, whereas the induced , position dependent mass functions pertaining to the oscillatory modes of three valence quarks in baryons are denoted by small letters  $m_\alpha$  .

In subsection 3-1-1 and section 4 below the (sub-) section numbering is taken over from ref. 14 – [14-2013] .

### 3-1-1 – Assembling elements of the QCD Lagrangean density – premises

We face the theoretical abstraction of QCD with  $N_{fl} = 6$  , representing strong interactions – adaptable to two or three light flavors u , d , s of quarks and antiquarks.  $\leftrightarrow$

quarks : color is counted in  $\pi^0 \rightarrow \gamma \gamma$

$\left( \begin{array}{l} \text{assuming global color- and} \\ \text{flavor-projections to commute} \end{array} \right)$  yet see ref. [21-2001]

spin and flavor are clearly seen in  $q\bar{q}$  and  $3q$  ,  $3\bar{q}$  spectroscopy

$\left( \begin{array}{l} \text{a pre-condition} \\ \text{to count color} \end{array} \right)$  .

$$\mathcal{L} = \left[ \bar{q}_{\dot{S}' f}^{\dot{c}'} \left\{ + W_\mu^r \left( \frac{1}{2} \lambda_r \right)_{c'\dot{c}} \frac{\vec{\partial}}{\partial} \delta_{c'\dot{c}} \right\} \gamma_{\dot{S}' S}^\mu q_{\dot{S} f}^c - m_f \bar{q}_{\dot{S} f}^{\dot{c}} q_{\dot{S} f}^c \right. \\ \left. - \frac{1}{4g^2} B^{\mu\nu r} B_{\mu\nu}^r + \Delta \mathcal{L} \right] \quad (45)$$

$W_\mu^r \equiv -v_\mu^r$  : for identification of convention for potentials

quarks :  $c' , c = 1, 2, 3$  color ,  $f = 1, \dots, 6$  flavor

$\dot{S}', S = 1, \dots, 4$  spin ,  $m_f$  mass

In eq. 45 the  $\mathcal{D}$  related gauge connection fields, where  $\mathcal{D} = \mathcal{D}(\mathcal{G})$  denotes a general, irreducible representation of the local gauge group  $\mathcal{G} = SU3_c$ ,

appear in the form appropriate for quarks :  $\mathcal{D} = \{ 3 \}$  , and antiquarks :  $\mathcal{D} = \{ \bar{3} \}$  respectively

$$\boxed{\begin{aligned} (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) &= W_\mu^r(x) (d_r)_{\alpha\beta} \\ \leftrightarrow \mathcal{W}_\mu(\mathcal{D}) &= -\mathcal{W}_\mu(\mathcal{D})^\dagger \\ d_r &= -d_r^\dagger = \frac{1}{i} J_r \in Lie(\mathcal{D}) ; [d_p, d_q] = f_{pqr} d_r \\ r, p, q &= 1, \dots, \dim \mathcal{G} ; \alpha, \beta = 1, \dots, \dim \mathcal{D} \end{aligned}} \quad (46)$$

For  $\mathcal{D}(SU3_c) = \{ 3(\bar{3}) \}$  the representation matrices become ( the Gell-Mann matrices [22-1964] )

$$\begin{aligned} (d_r(3) &= \frac{1}{i} \frac{1}{2} \lambda_r)_{\alpha\beta} ; r = 1, \dots, 8 \\ (\alpha, \beta) &\leftrightarrow (c', \dot{c}) = 1, \dots, 3 ; d_r(\bar{3}) = \bar{d}_r(3) \end{aligned}$$

$$\text{with the conventional normalization conditions : } -tr d^r d^s = \frac{1}{2} \delta^{rs} \quad (47)$$

The quantity proportional to the gauge potentials  $W_\mu^r$  for the  $\bar{q}q$  in eq. 45 is thus identified as

$$[W_\mu^r(\frac{1}{2}\lambda_r)_{c'\dot{c}} = i(\mathcal{W}_\mu(\mathcal{D} = \{ 3 \}))_{c'\dot{c}}](x) \quad (48)$$

Here we postpone the discussion of complete connections and extend the QCD Lagrangean density to include the term quadratic in the field strengths  $B_{\mu\nu}^r$  and  $\Delta \mathcal{L}$  in eq. 45, in Fermi gauges.

$$\begin{aligned} \text{gauge bosons : } \mathcal{L}_B &= -\frac{1}{4g^2} B^{\mu\nu r} B_{\mu\nu}^r \\ B_{\mu\nu}^r &= \partial_\mu W_\nu^r - \partial_\nu W_\mu^r + f_{rst} W_\mu^s W_\nu^t \longleftarrow (W_\mu^r \equiv -v_\mu^r) \\ r, s, t &= 1, \dots, \dim(\mathcal{G} = SU3_c) = 8 \\ \text{Lie algebra labels, } [\frac{1}{2}\lambda^r, \frac{1}{2}\lambda^s] &= i f_{rst} \frac{1}{2}\lambda^t \\ \text{perturbative rescaling :} \\ W_\mu^r &= g W_{\mu \text{ pert}}^r, B_{\mu\nu}^r = g B_{\mu\nu \text{ pert}}^r \end{aligned} \quad (49)$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.

Completing  $\Delta \mathcal{L}$  in Fermi gauges

$$\begin{aligned} \Delta \mathcal{L} &= \left\{ -\frac{1}{2\eta g^2} (\partial_\mu W^{\mu r})^2 \right\} ; \eta : \text{gauge parameter} \\ \text{ghost fermion fields : } c, \bar{c} ; (D_\mu c)^r &= \partial_\mu c^r + f_{rst} W_\mu^s c^t \\ \text{gauge fixing constraint : } C^r &= \partial_\mu W^{\mu r} \end{aligned} \quad (50)$$

### 3-1-2a – Gauge boson binary bilocal and adjoint ( here octet- ) string operators

One goal is, to identify – not just some candidate resonance – gluonic mesons, binary and higher modes, and to relate them to the base quantities within QCD . Here we follow ref. [17-2011] .

$$\begin{aligned} B_{[\mu_1 \nu_1], [\mu_2 \nu_2]}(x_1, x_2) &= \\ &= B_{[\mu_1 \nu_1]}^r(x_1) U(x_1, r; x_2, s) B_{[\mu_2 \nu_2]}^s(x_2) \end{aligned} \quad (51)$$

$r, s, \dots = 1, \dots, 8$  ; no flavor but spin

$B_{[\mu \nu]}^r(x)$  denote the local color octet of field strengths.  
The quantity  $U(x, r; y, s)$  in eq. (51) denotes the octet string operator,  
i. e. the path ordered exponential over a straight line path  $\mathcal{C}$  from y to x

$$\begin{aligned} U(x, r; y, s) &= P \exp \left( \int_y^x \Big|_{\mathcal{C}} dz^\mu \frac{1}{i} v_\mu^t(z) \mathcal{F}_t \right) \Big|_{rs} \\ &= P \exp \left( - \int_y^x \Big|_{\mathcal{C}} dz^\mu W_\mu^t(z) \left( \frac{1}{i} \mathcal{F}_t \right) \right) \Big|_{rs} \\ (\mathcal{F}_t)_{rs} &= i f_{rst} ; (ad_t)_{rs} = \frac{1}{i} (\mathcal{F}_t)_{rs} = f_{rst} \end{aligned} \quad (52)$$

The path ordered exponential as a matrix function of the argument is to be performed before the matrix elements, denoted  $|..$  in eq. 52 , are taken.  
The local limit becomes

$$\begin{aligned} B_{[\mu_1 \nu_1], [\mu_2 \nu_2]}(x_1 = x_2 = x) &= \\ &= (:) B_{[\mu_1 \nu_1]}^r(x) B_{[\mu_2 \nu_2]}^s(x) (:); \text{ no flavor but spin} \end{aligned} \quad (53)$$

### 2-3 – $\bar{q} q$ bilinears and triplet-string operators

$$\begin{aligned} B_{[\mathcal{A} f_1, \mathcal{B} f_2]}^q(x_1, x_2) &= \\ &= \bar{q}_{\mathcal{B} f_2}^{\dot{c}_1}(x_1) U(x_1, c_1; x_2, \dot{c}_2) q_{\mathcal{A} f_1}^c(x_2) \end{aligned}$$

flavor and spin

$$\begin{aligned} U(x, c_1; y, \dot{c}_2) &= P \exp \left( \int_y^x \Big|_{\mathcal{C}} dz^\mu \frac{1}{i} v_\mu^t(z) \frac{1}{2} \lambda_t \right) \Big|_{c_1 \dot{c}_2} \\ &= P \exp \left( - \int_y^x \Big|_{\mathcal{C}} dz^\mu W_\mu^t(z) \left( \frac{1}{i} \frac{1}{2} \lambda_t \right) \right) \Big|_{c_1 \dot{c}_2} \end{aligned} \quad (54)$$

with the local limit

$$B_{[\dot{\mathcal{B}} f_2, \mathcal{A} f_1]}^q(x_1 = x_2 = x) = (:) \bar{q}_{\dot{\mathcal{B}} f_2}^{\dot{c}}(x) q_{\mathcal{A} f_1}^c(x) (:) \quad (55)$$



The symbols  $(:)$  in eqs. 53 and 55 should indicate that normal ordering of regulating the local limits is required and further that such normal ordering is *not* unique, related to Bogoliubov transformations, and dependent on quark masses in the case of the  $\bar{q}q$  bilinears.

#### 2-4 – Connection and curvature - forms preparing the ensuing analysis of regularity conditions

Lets begin this (sub-)section rewriting the bilocal (formally) unitary operators forming the gauge connection dependent octet- ( eq. 52 ) and triplet ( eq. 54 ) QCD strings, substituting an equivalent, matrix oriented notation

$$\begin{aligned}
\text{octet string} : U(x, r; y, s) &= P \exp \left( - \int_y^x \Big|_C dz^\mu W_\mu^t(z) \left( \frac{1}{i} \mathcal{F}_t \right) \right) \Big|_{rs} \\
\text{triplet string} : U(x, c_1; y, \dot{c}_2) &= P \exp \left( - \int_y^x \Big|_C dz^\mu W_\mu^t(z) \left( \frac{1}{i} \frac{1}{2} \lambda_t \right) \right) \Big|_{c_1 \dot{c}_2} \\
&\text{with the substitutions } \longrightarrow \\
\text{octet string} : U(x, r; y, s) &\rightarrow \left( U \left( x \stackrel{C}{\leftarrow} y \right) \right)_{rs} \\
\text{triplet string} : U(x, c_1; y, \dot{c}_2) &\rightarrow \left( U \left( x \stackrel{C}{\leftarrow} y \right) \right)_{c_1 \dot{c}_2} \left. \vphantom{\begin{matrix} \text{octet string} \\ \text{triplet string} \end{matrix}} \right\} \begin{matrix} \rightarrow \mathcal{U}(x, C, y; \mathcal{D})_{\alpha\beta} \\ \in \mathcal{D}(\mathcal{G}) \end{matrix} \\
&\text{with } \mathcal{G} = \text{simple compact gauge group;} \\
&\mathcal{D} : \text{general irreducible representation of } \mathcal{G}
\end{aligned} \tag{56}$$

Here  $\mathcal{G} = SU3_c$  and  $\mathcal{D}$  is the octet-, triplet representation for the respective QCD  $\mathcal{D}$ -strings.

Further let us consider matrix valued connection 1-forms, which define the bilocal matrix valued operators  $(U(x, C, y; \mathcal{D}))_{\alpha\beta} \in \mathcal{D}(\mathcal{G})$  as given in eq. 56. To this end the form of octet and triplet strings in eq. 56 is repeated below

$$\begin{aligned}
\text{octet string} : U(x, r; y, s) &= P \exp \left( - \int_y^x \Big|_C dz^\mu W_\mu^t(z) \left( \frac{1}{i} \mathcal{F}_t \right) \right) \Big|_{rs} \\
\text{triplet string} : U(x, c_1; y, \dot{c}_2) &= P \exp \left( - \int_y^x \Big|_C dz^\mu W_\mu^t(z) \left( \frac{1}{i} \frac{1}{2} \lambda_t \right) \right) \Big|_{c_1 \dot{c}_2}
\end{aligned} \tag{57}$$

The two matrices in brackets to the right of the integrand expressions in eq. 57 form an antihermitian basis of the Lie algebra representation  $Lie(\mathcal{D})$  for  $\mathcal{D} = \text{adjoint}$  and  $\mathcal{D} = \text{triplet}$  representations of  $\mathcal{G} = SU3_c$  respectively

$$\begin{aligned}
d_t \equiv d_t(\mathcal{D}) \leftrightarrow (d_t)_{\alpha\beta} &= \begin{cases} \left( \frac{1}{i} \mathcal{F}_t \right)_{rs} & \text{for } Lie(\mathcal{D}) = \text{adjoint} \\ \left( \frac{1}{i} \frac{1}{2} \lambda_t \right)_{c_1 \dot{c}_2} & \text{for } Lie(\mathcal{D}) = \text{triplet} \end{cases} \\
d_t &= -d_t^\dagger; \quad t = 1, \dots, \dim \mathcal{G}; \quad \alpha, \beta = 1, \dots, \dim \mathcal{D} \text{ for general } \mathcal{D}
\end{aligned} \tag{58}$$

From eqs. 57 and 58 we construct a – hopefully – consistent notation as appropriate for matrix valued  $\mathcal{D}$  connections, 1-forms and strings, as well as derived 2- and higher forms. First eq. 58 is subject to the ( matrix- ) commutation relations

$$\begin{aligned} [d_r, d_s] &= f_{rst} d_t ; \forall \mathcal{D}(\mathcal{G}) ; r, s, t = 1, \dots, \dim \mathcal{G} \longrightarrow \\ (d_t (\mathcal{D} = \text{adjoint representation}))_{sr} &= (ad_t)_{sr} = f_{str} : \text{independent of } \mathcal{D} \\ f_{str} &: \text{totally antisymmetric, real structure constants of } Lie(\mathcal{G}) \end{aligned} \quad (59)$$

In physics the antihermitian matrix code with respect to the representations  $Lie(\mathcal{D})$  is ( mostly ) replaced by the hermitian one <sup>1</sup>

$$\begin{aligned} (d_t \equiv \frac{1}{i} h_t) (Lie(\mathcal{D}))|_{\alpha\beta} ; h_t &= h_t^\dagger ; [h_r, h_s] = i f_{rst} h_t \\ \alpha, \beta &= 1, \dots, \dim \mathcal{D} \end{aligned} \quad (60)$$

Eq. 59 serves to define matrix valued connections built from a basis of  $Lie(\mathcal{D})$  representation matrices as defined in eq. 59 for general irreducible representations  $\mathcal{D}(\mathcal{G})$  denoted  $\mathcal{W}_\mu(z, \mathcal{D})$

$$\begin{aligned} \mathcal{W}_\mu(z, \mathcal{D})|_{\alpha\beta} &= W_\mu^r(z) (d_r)_{\alpha\beta} Lie(\mathcal{D}) ; \left[ \begin{array}{l} r = 1, \dots, \dim(\mathcal{G}) \\ \alpha, \beta = 1, \dots, \dim(\mathcal{D}) \end{array} \right] \\ \mathcal{W}_\mu(z, \mathcal{D})|_{\alpha\beta} &\longrightarrow \mathcal{W}_\mu \text{ for compact matrix notation} \end{aligned} \quad (61)$$

In the following it is to be understood, that  $\mathcal{W}_\mu$  is extended to a general collection of representations  $\bigcup \mathcal{D}$  – thought to be carried by real and spurious spin  $\frac{1}{2}$  fields – care being taken that asymptotic freedom in the ultraviolet is not upset.

From eq. 61 we define the associated matrix valued connection 1-form displayed alongside the base definition repeated from eq. 61 in eq. 62 below

$$\begin{aligned} \mathcal{W}^{(1)}(z, \mathcal{D})|_{\alpha\beta} &= d z^\mu \mathcal{W}_\mu(z, \mathcal{D})|_{\alpha\beta} \longrightarrow \mathcal{W}^{(1)} \\ \mathcal{W}_\mu(z, \mathcal{D})|_{\alpha\beta} &= W_\mu^r(z) (d_r)_{\alpha\beta} Lie(\mathcal{D}) \longrightarrow \mathcal{W}_\mu \end{aligned} \quad (62)$$

and the matrix valued field-strength tensor

$$\mathcal{W}_{\mu\nu}(z, \mathcal{D})|_{\alpha\beta} = \left\{ \begin{array}{l} \partial_\mu \mathcal{W}_\nu(z, \mathcal{D}) - \partial_\nu \mathcal{W}_\mu(z, \mathcal{D}) + \\ + [\mathcal{W}_\mu(z, \mathcal{D}), \mathcal{W}_\nu(z, \mathcal{D})] \end{array} \right\}_{\alpha\beta} \longrightarrow \mathcal{W}_{\mu\nu} \quad (63)$$

---

<sup>1</sup> A ( partial ) collection of historical and textbook references to the topics pertaining to 'Continuous transformation groups and differential geometry' is presented under R-H references and labelled by the symbols 1H, 2H ... in refs. 14 – [14-2013] and 16 – [16-2012].

together with their associated curvature 2-form

$$\begin{aligned}
\mathcal{W}^{(2)}(z, \mathcal{D})|_{\alpha\beta} &= \frac{1}{2} d z^\mu \wedge d z^\mu \mathcal{W}_{\mu\nu}(z, \mathcal{D})|_{\alpha\beta} \longrightarrow \mathcal{W}^{(2)} \\
\mathcal{W}_{\mu\nu}(z, \mathcal{D})|_{\alpha\beta} &= \left\{ \begin{aligned} &\partial_\mu \mathcal{W}_\nu(z, \mathcal{D}) - \partial_\nu \mathcal{W}_\mu(z, \mathcal{D}) + \\ &+ [\mathcal{W}_\mu(z, \mathcal{D}), \mathcal{W}_\nu(z, \mathcal{D})] \end{aligned} \right\}_{\alpha\beta} \longrightarrow \mathcal{W}_{\mu\nu} \\
&= W_{\mu\nu}^r(z) (d_r)_{\alpha\beta} \text{ Lie } (\mathcal{D})
\end{aligned}$$


---

$$W_{\mu\nu}^r = \partial_\mu W_\nu^r - \partial_\nu W_\mu^r + f_{rst} W_\mu^s W_\nu^t ; \text{ independent of } \mathcal{D}$$

(64)

Two remarks are in place here

- 1) In order to distinguish field strengths from potentials ( connections ) the following equivalent but different notations for the field strength shall be used

$$\mathcal{W}_{\mu\nu} \equiv \mathcal{B}_{\mu\nu} ; \mathcal{W}^{(2)} \equiv \mathcal{B}^{(2)} ; W_{\mu\nu}^r \equiv B_{\mu\nu}^r \quad (65)$$

- 2) From the last relation in eq. 64 it may appear redundant to extend connections and curvatures to matrix valued form with respect to a wide collection of irreducible representations  $\mathcal{D}(\mathcal{G})$ . This however is tantamount to neglecting nontrivial global regularity conditions in the infrared .

We end this subsection ( 2-4 ) displaying the bilocal ( parallel transport- ) operators defined in eq. 56 using the shorthand notation in eq. 64

$$\begin{aligned}
(U(x, C, y; \mathcal{D}))_{\alpha\beta} &= P \exp \left( - \int_y^x \Big|_C \mathcal{W}^{(1)}(z, \mathcal{D}) \right) \Big|_{\alpha\beta} \\
\downarrow & \qquad \qquad \qquad \downarrow \\
U(x, C, y) &= P \exp \left( - \int_y^x \Big|_C \mathcal{W}^{(1)} \right) [\text{for } (\cup \mathcal{D})]
\end{aligned} \quad (66)$$

## 2-5 – The U1- or singlet axial current anomaly

The U1-axial central anomaly involves the local chiral current projections from  $B_{[\dot{B} f_2, A f_1]}^q(x)$  in eq. 55

$$\begin{aligned}
(j_\mu^\pm)_{f_2 f_1}(x) &= B_{[\dot{B} f_2, A f_1]}^q(x) (\gamma_\mu \frac{1}{2} (\P \pm \gamma_5))_{B\dot{A}} \\
&= (\cdot) \bar{q}_{\dot{f}_2} \gamma_\mu^\pm q_{f_1}^c(x) (\cdot) \\
\gamma_5 &= \gamma_5 R = \frac{1}{i} \gamma_0 \gamma_1 \gamma_2 \gamma_3 ; \gamma_\mu^\pm = \gamma_\mu \frac{1}{2} (\P \pm \gamma_5)
\end{aligned} \quad (67)$$

The equations of motion for the fermion fields are *and superficially imply* (upon  $f_1 \leftrightarrow f_2$ )

$$\begin{aligned}
\partial q_{f_2}^c &= \frac{1}{i} \left( \not{p}^c \not{c}' + \delta^{cc'} m_{f_2} \right) q_{f_2}^{c'} \\
\overline{q}_{f_1}^{\dot{c}} \overleftarrow{\partial} &= \overline{q}_{f_1}^{\dot{c}'} \frac{1}{i} \left( -\not{p}^{c'\dot{c}} - \delta^{c'\dot{c}} m_{f_1} \right) ; \text{ no sums over } f_1, f_2 \rightarrow \\
\partial^\mu (j_\mu^\pm)_{f_1 f_2} &= \frac{1}{2i} \left( (m_{f_2} - m_{f_1}) S_{f_1 f_2} \mp (m_{f_2} + m_{f_1}) P_{f_1 f_2} \right) \\
S_{f_1 f_2} &= (:) \overline{q}_{f_1}^{\dot{c}} q_{f_2}^c (:), P_{f_1 f_2} = (:) \overline{q}_{f_1}^{\dot{c}} \gamma_5 q_{f_2}^c (:)
\end{aligned} \tag{68}$$

In eq. 68  $m_f$  denotes the *real, nonnegative* quark mass for flavor  $f$ .  
From eq. 68 the relations for vector and axial vector currents *superficially* follow

$$\begin{aligned}
(j_\mu)_{f_1 f_2} &= (j_\mu^+)_{f_1 f_2} + (j_\mu^-)_{f_1 f_2} \\
(j_\mu^5)_{f_1 f_2} &= (j_\mu^+)_{f_1 f_2} - (j_\mu^-)_{f_1 f_2} \\
\partial^\mu (j_\mu)_{f_1 f_2} &= \frac{1}{i} (m_{f_2} - m_{f_1}) S_{f_1 f_2} \\
\partial^\mu (j_\mu^5)_{f_1 f_2} &= (m_{f_2} + m_{f_1}) i P_{f_1 f_2}
\end{aligned} \tag{69}$$

As it follows from the original derivation by Adler and Bell and Jackiw [23-1969] in QED, the vector current Ward identities in eq. 69 can be implemented also in QCD, leaving the axial current ones reduced to the flavor non-singlet case, leaving the U1 axial current divergent anomalous

$$\begin{aligned}
\partial^\mu (j_\mu)_{f_1 f_2} &= \frac{1}{i} (m_{f_2} - m_{f_1}) S_{f_1 f_2} \quad \checkmark \\
\left\{ \begin{matrix} j_\mu^5 \\ P \end{matrix} \right\}_{f_1 f_2}^{NS} &= \left\{ \begin{matrix} j_\mu^5 \\ P \end{matrix} \right\}_{f_1 f_2} - \frac{1}{N_{fl}} \delta_{f_1 f_2} \sum_f \left\{ \begin{matrix} j_\mu^5 \\ P \end{matrix} \right\}_{f f}
\end{aligned} \tag{70}$$

and similarly

$$\left\{ \begin{matrix} j_\mu^5 \\ P \end{matrix} \right\}_{f_1 f_2}^S = \sum_f \left\{ \begin{matrix} j_\mu^5 \\ P \end{matrix} \right\}_{f f} \tag{71}$$

## 2-6 – Quark masses and splittings : $m_f$ and $\Delta m_f = m_f - \langle m \rangle$

In the subtitle above  $\langle m \rangle$  stands for the mean quark mass

$$\langle m \rangle = \frac{1}{N_{fl}} \sum_f m_f \tag{72}$$

The identities for vector currents in eqs. 69 and 70 can be extended separating the contributions proportional to  $\Delta m_f$  and  $\langle m \rangle$

$$\begin{aligned}
\partial^\mu (j_\mu)_{f_1 f_2} &= \frac{1}{i} (\Delta m_{f_2} - \Delta m_{f_1}) S_{f_1 f_2} \checkmark \\
\partial^\mu (j_\mu^5)_{f_1 f_2}^{NS} &= (\Delta m_{f_2} + \Delta m_{f_1}) i P_{f_1 f_2}^{NS} \checkmark \\
\partial^\mu (j_\mu^5)_{f_1 f_2}^S &= 2 \langle m \rangle i P^S \not\checkmark [\longrightarrow + \delta_5] \\
\delta_5 &= (2 N_{fl}) \frac{1}{32\pi^2} B_{\mu\nu}^r \tilde{B}^{\mu\nu r} \Big|_{\rightarrow ren.gr.inv} ; \tilde{B}_{\mu\nu}^r = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} B^{\sigma\tau r}
\end{aligned} \tag{73}$$

We shall return to the question of how the local operator  $ch_2(B) \equiv \frac{1}{32\pi^2} (:B_{\mu\nu}^r \tilde{B}^{\mu\nu r}:)$  is to be normalized and rendered renormalization group invariant [24-1991]. Here we just assume this to have been achieved and denote the U1-axial anomaly, the first of the central two, in its general form

$$\begin{aligned}
&(\text{eq. 73}) \\
&\left\{ \partial^\mu (j_\mu^5)^S = 2 \langle m \rangle i P^S + \delta_5 \right\} (x) \\
&\delta_5 = (2 N_{fl}) \frac{1}{32\pi^2} (:B_{\mu\nu}^r \tilde{B}^{\mu\nu r}:) \Big|_{\rightarrow ren.gr.inv}
\end{aligned} \tag{74}$$

From here it is conceptually clear how the scale- (or trace-) anomaly arises but strictly within QCD. The renormalizability of a field theory in the limit of uncurved space-time gives rise to a local, symmetric and *conserved* energy momentum tensor, implying exact Poincaré invariance

$$\begin{aligned}
&\{ \vartheta_{\mu\nu} = \vartheta_{\nu\mu} \} (x) \\
&\partial^\nu \vartheta_{\mu\nu} = 0
\end{aligned} \tag{75}$$

In connection with the normal ordering questions it is important to admit in the precise form of the energy momentum tensor a nontrivial vacuum expected value, which in view of exact Poincaré invariance must be of the form

$$\begin{aligned}
&\langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle = \frac{1}{4} \eta_{\mu\nu} \tau \\
&\left\{ \begin{array}{c} \eta_{\mu\nu} = diag(1, -1, -1, -1) \\ \tau \end{array} \right\} \text{ independent of } x \longrightarrow \\
&\Delta \vartheta_{\mu\nu}(x) = \vartheta_{\mu\nu}(x) - \langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle \times \left\{ \begin{array}{c} \hat{\mathbb{1}} \\ \text{or } |\Omega\rangle \langle \Omega| \end{array} \right. \\
&\text{with } \partial^\nu \Delta \vartheta_{\mu\nu}(x) = 0 ; \langle \Omega | \Delta \vartheta_{\mu\nu}(x) | \Omega \rangle = 0
\end{aligned} \tag{76}$$

In eq. 76  $\hat{\mathbb{1}}$  denotes the unit operator in the entire Hilbert space of states, while  $P_\Omega = |\Omega\rangle \langle \Omega|$  stands for the projector on the ground state.

---

<sup>2</sup>  $\delta_5$  was, as far as I know, introduced by Murray Gell-Mann in lectures  $\sim 1970$  in Hawaii.

Furthermore from the two local, *conserved* tensors in eq. 76 only  $\Delta \vartheta_{\mu \nu}(x)$  with vanishing vacuum expected value is acceptable as representing the conserved 4 momentum *operators* in the integral form

$$\hat{P}_{\mu} = \int_t d^3 x \Delta \vartheta_{\mu 0}(t, \vec{x}) \quad (77)$$

All these arguments *notwithstanding* to subtract any eventual vacuum expected values of local operators, often put forward as mathematical prerequisites, it is wise *not to do so* in the presence of spontaneous parameters, the dynamical origin of spontaneous symmetry breaking, e.g. chiral symmetries in the limit or neighbourhood of some  $m_f \rightarrow 0$ .

Using the (classical) equations of motion pertaining to the Lagrangean in eqs. 44 - 45

$$\begin{aligned} (D_{\nu} B^{\mu\nu})^r &= j^{\mu r}(\bar{q}, q); B \rightarrow B_{pert} \\ (D_{\varrho} B^{\mu\nu})^r &= \partial_{\varrho} B^{\mu\nu r} + f_{rst} W_{\varrho}^s B^{\mu\nu t} \\ j_{\mu}^r(\bar{q}, q) &= g \bar{q}_{\dot{A}f} (\gamma_{\mu})_{\dot{A}B} \frac{1}{2} (\lambda^r)_{c\dot{c}'} q_{\dot{A}f}^{c'} \\ i(\gamma^{\mu} D_{\mu} q)_{\dot{A}f}^c &= m_f q_{\dot{A}f}^c \text{ and } q \rightarrow \bar{q} \\ (D_{\mu} q)_{\dot{A}f}^c &= [\partial_{\mu} \delta_{c\dot{c}'} + \frac{1}{i} W_{\mu}^t \frac{1}{2} (\lambda^t)_{c\dot{c}'}] q_{\dot{A}f}^{c'} \end{aligned} \quad (78)$$

---


$$W_{\mu}^r \equiv -v_{\mu}^r = g(W_{\mu}^r)_{pert} \equiv -g(v_{\mu}^r)_{pert}$$

the associated form of the energy momentum becomes

$$\vartheta_{\mu\nu}^{(cl)} = \left[ \frac{1}{4g^2} [B_{\mu\varrho}^t B_{\nu}^{\varrho t} - \frac{1}{4} \eta_{\mu\nu} B_{\sigma\varrho}^t B^{\varrho\sigma t}] + \right. \quad (79)$$

$$\left. + \frac{1}{2} \left[ \bar{q}_f \gamma_{\mu} \frac{i}{2} \vec{D}_{\nu} q_f + \mu \leftrightarrow \nu \right] \right]$$

and using once more the fermion part of the equations of motion the trace of the classical energy momentum tensor becomes

$$\begin{aligned} \vartheta_{\mu}^{\mu (cl)} &= \sum_f m_f S_{ff} \\ S_{f_1 f_2} &= (\cdot) \bar{q}_{\dot{f}_1} q_{f_2}^c (\cdot) \end{aligned} \quad (80)$$

## 2-7 – The scale- or trace- anomaly

From the classical soft fermionic contribution to the trace of the energy momentum tensor there is a clear conjecture, also by Murray Gell-Mann, of the anomalous contribution, which subsequently became the scale- or trace- anomaly within QCD

$$\begin{aligned} \vartheta_{\mu}^{\mu} &= \sum_f m_f S_{ff} + \delta_0 \\ \delta_0 &= -(-2\beta(g)/g^3) \left[ \frac{1}{4} (\cdot) B_{\mu\nu}^t B^{\mu\nu t} (\cdot) \right] \rightarrow_{ren.gr.inv} \end{aligned} \quad (81)$$

## 2-8 – The two central anomalies alongside : scale- or trace- and U1-axial anomaly

We collect the two anomalous identities in eqs. 81 and 63

$$\begin{aligned}
& \left\{ \vartheta^\mu{}_\mu = \sum_f m_f S_{\dot{f}f} + \delta_0 \right\} (x) \\
& \left\{ \partial^\mu (j_\mu^5)^S = 2 \langle m \rangle i P^S + \delta_5 \right\} (x) \\
& \delta_0 = - \left( -2 \beta(g) / g^3 \right) \left[ \frac{1}{4} (:) B_{\mu\nu}^t B^{\mu\nu t} (:) \right] \rightarrow_{ren.gr.inv} \\
& \delta_5 = (2 N_{fl}) \frac{1}{8\pi^2} \left[ \frac{1}{4} (:) B_{\mu\nu}^t \tilde{B}^{\mu\nu t} (:) \right] \rightarrow_{ren.gr.inv} \\
& \hline
& -\beta/g^3 = \frac{1}{16\pi^2} b_0 + O(X) ; X = g^2 / (16\pi^2) \\
& \beta(g) : \text{Callan-Symanzik rescaling function in QCD}
\end{aligned} \tag{82}$$

The qualification 'central' for the anomalies in eq. 82 stands for the property that in rendering the square coupling constant and the associated  $\vartheta$  – parameter in the gauge boson *renormalized* Lagrangean density  $x$  dependent

$$\begin{aligned}
\mathcal{L}_{g.b.} &= -\frac{1}{g^2} \frac{1}{4} (:) B_{\mu\nu}^t B^{\mu\nu t} (:) + \vartheta \frac{1}{8\pi^2} \frac{1}{4} (:) B_{\mu\nu}^t \tilde{B}^{\mu\nu t} \longrightarrow \\
&g^2 \rightarrow g^2(x) ; \vartheta \rightarrow \vartheta(x)
\end{aligned} \tag{83}$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions as external sources for the scalar and pseudoscalar local field strength bilinears

$$\frac{1}{4} (:) B_{\mu\nu}^t B^{\mu\nu t} (:) , \frac{1}{4} (:) B_{\mu\nu}^t \tilde{B}^{\mu\nu t} \tag{84}$$

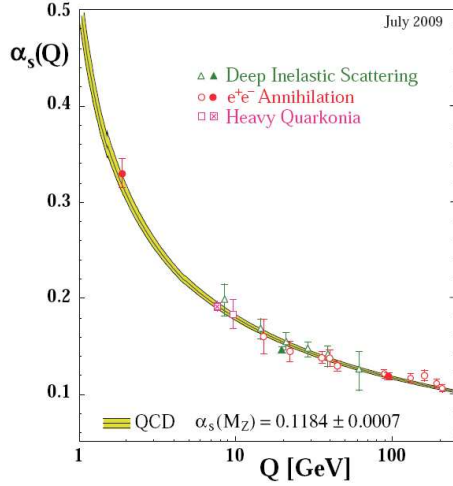
We will use the following definitions relative to the rescaling function  $\beta$

$$\begin{aligned}
& -\beta/g = X B(X) ; B(X) = b_0 A(X) \\
& B(X) \sim \sum_{n=0}^{\infty} b_n X^n , A(X) \sim \sum_{n=0}^{\infty} a_n X^n \\
& \kappa = g^2 / (16\pi^2) \text{ generic } \longrightarrow X, Y \\
& b_0 = \frac{1}{3} (33 - 2 N_{fl}) , a_0 = 1 , a_n = b_n / b_0 \\
& b_1 = \frac{2}{3} (153 - 19 N_{fl}) \\
& b_2 = \frac{1}{54} \left( 77139 - 15099 N_{fl} + 325 N_{fl}^2 \right) \\
& b_3 \sim 29243 - 6946.3 N_{fl} + 405.089 N_{fl}^2 + 1.49931 N_{fl}^3
\end{aligned} \tag{85}$$

References for this section ( 2 - premises ) are presented in five (partial) collections in ref. 16 – [16-2012] :

- 1 : (R) directly related to the two central anomalies
- 2 : (rBsquare) establishing the one renormalization group invariant quantity of dimension  $[M^4]$

- 3 : (r-sp-1) a recent paper by Guido Altarelli and references cited therein
- 4 : (r-A2x) a selection of papers and textbooks for the entire realm of QCD
- 5 : (r-condx) : Condensation phenomena and field theory realizations



**Fig. A21 :**  $\alpha_s(Q) = 4\pi \kappa_{\bar{\mu}=Q}$  from ref. [25-2009] .

This ends section 2 – premises

#### 4 – Ideas forging and foregoing - the dynamics of genuinely oscillatory modes [15-1980]

It is worth noting , that Erwin Schrödinger turned to the discussion of oscillatory modes of single- and by reduction of c.m. coordinates – of a pair mode of oscillatory motion , in the last ( 4th ) paper in ref. [26-1926] .

However in the above paper he ( E.S. ) makes the assumption , that associated forces arise from the exchange of a photon , i.e. involve a local electromagnetic exchange interaction as giving rise to an equally local second order wave equation , responsible for oscillatory (pair-) modes . This is incorrect , contrary to the structure embedded in QCD , up to the present incomplete level of completion , which remains a future task .

Thus we concentrate on the present topic and lay out the ideas in ref. [15-1980] .

We relate the bond structure of quark-antiquark ( meson ) and  $N$ – quark ( baryon ) systems, subject to an  $SU(N)$  unbroken color gauge group , to the long-range dynamics involving the oscillatory modes in the phase space



of the center of mass position and momentum variables [  $N \equiv N_c \rightarrow 3$  ] to be clear.

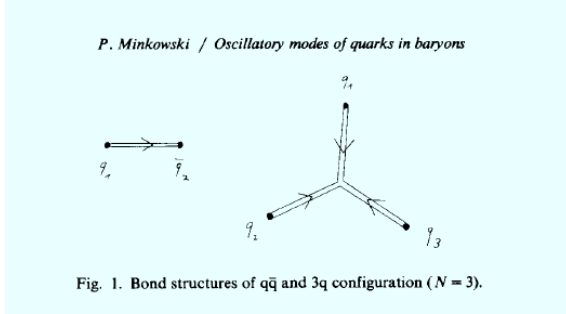
These canonical barycentric 3-vector variables are shown in eq. 86

$$\begin{aligned}
\vec{\pi}_1 &= \frac{1}{\sqrt{2}}(\vec{p}_1 - \vec{p}_2) & , & & \vec{z}_1 &= \frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2) \\
\vec{\pi}_2 &= \frac{1}{\sqrt{6}}(\vec{p}_1 + \vec{p}_2 - 2\vec{p}_3) & , & & \vec{z}_2 &= \frac{1}{\sqrt{6}}(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3) \\
&\vdots & & & \vdots & \\
&\vdots & & & \vdots & \\
\vec{\pi}_\nu &= (\nu(\nu+1))^{-1/2} \begin{pmatrix} \sum_{\alpha=1}^{\nu} \vec{p}_\alpha \\ -\nu \vec{p}_{\nu+1} \end{pmatrix} , & \vec{z}_\nu &= (\nu(\nu+1))^{-1/2} \begin{pmatrix} \sum_{\alpha=1}^{\nu} \vec{x}_\alpha \\ -\nu \vec{x}_{\nu+1} \end{pmatrix} \\
&\vdots & & & \vdots & \\
&\vdots & & & \vdots & \\
\vec{\pi}_{N-1} &= \cdots & , & & \vec{z}_{N-1} &= \cdots
\end{aligned}$$


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$$\vec{\pi}_N = N^{-1/2} \sum_{\alpha=1}^N \vec{p}_\alpha \rightarrow 0 \quad , \quad \vec{z}_N = N^{-1/2} \sum_{\alpha=1}^N \vec{x}_\alpha \rightarrow 0 \quad (86)$$

The last line in eq. 86 refers to c.m. momentum and position .  
The bond structures of  $q\bar{q}$  and  $3q$  configurations are shown in figure 1 - I , repeated below



**Fig. 1 - I : Bond structures of  $q\bar{q}$  and  $3q$  configurations**  
 $(N = 3) \longleftrightarrow$

Of course we are interested in  $N = 3$  but one goal of this investigation is to shed light on the  $N$  dependence of the ratio of baryonic to mesonic inverse Regge slopes

$$\Lambda_N / \Lambda = 1 \text{ for vanishing quark masses } M_\alpha \rightarrow 0 ; \alpha = 1, \dots, N \quad (87)$$

$$\begin{aligned}\Delta \mathcal{M}_{baryon}^2 &= 2 \Lambda_N \sum_{\alpha=1}^{3N-3} \Delta \nu_{\alpha} \\ \Delta \mathcal{M}_{meson}^2 &= 2 \Lambda \sum_{\alpha=1}^3 \Delta \nu_{\alpha}\end{aligned}\quad \Delta \nu_{\alpha} = 0, \pm 1, \pm 2, \dots \quad (88)$$

In eq. 87 quark masses are denoted  $M_{\alpha}$ ;  $\alpha$ : quark flavor, as throughout section 4, in order to distinguish them from – the oscillator state configuration space variable dependent masses – denoted  $m, \bar{m} \dots$ .

The connection of this ratio to the intermediary range quark-antiquark potential – mainly studied for the  $c\bar{c}$  charmonium (binding) spectrum – determined by the conditions

$$\text{Min}_{\alpha=1,\dots,N} M_{\alpha} \ll |V_{NR}(z_{\beta})| \ll \text{Min}_{\gamma=1,\dots,N-1} |z_{\gamma}|^{-1} \quad (89)$$

We reformulate as starting point the snowball effect for a  $q\bar{q}$  equal mass pair, described (in the c.m. system) by the Lagrangean

$$\begin{aligned}\mathcal{L}_{(2)} &= -m_1 (1 - v_1^2)^{-1/2} - m_2 (1 - v_2^2)^{-1/2} \\ &= -2m (1 - v^2)^{-1/2} \\ \vec{v}_j &= \dot{\vec{x}}_j; j = 1, 2; m = m(z) \text{ for (just here) } M_{q_1} = M_{q_2} = 0\end{aligned}\quad (90)$$

Eq. 90 is only valid in the c.m. frame, where the analog of energy conservation takes the form

$$\begin{aligned}\mathcal{H}_{(2)} &= \vec{v} \mathcal{L}_{(2)}, \vec{v} = \mathcal{L}_{(2)} = \frac{2m}{\sqrt{1 - v^2}} \\ (\vec{p})_1 - (\vec{p})_2 &= 2\vec{p}_{c.m.} = \mathcal{L}_{(2)}, \vec{v} = \mathcal{H}_{(2)} \vec{v} \\ (\mathcal{H}_{(2)})^2 v^2 &= (\mathcal{H}_{(2)})^2 - 4m^2 = 4p_{c.m.}^2\end{aligned}\quad (91)$$

Eqs. 90 and 91 are understood as approximations for large distances. They can be interpreted classically or quantum mechanically.

We note that in the discussion ongoing of  $q\bar{q}$  oscillator modes, we do not use the orthogonal normalization as displayed in eq. 86. This is so because other conventions had been used before, as for myself to the year 1976, while working at Caltech. The definitions used are shown in the next equation

$$\begin{aligned}\bar{m} &= 2m, -\Delta_z + \bar{m}^2(z) = H_{(2)}^2, z \rightarrow \vec{y} \\ (\mathcal{H}_{(2)} \vec{v}) \cdot &= \mathcal{H}_{(2)} \frac{1}{2} \ddot{\vec{y}} = -\frac{1}{\mathcal{H}_{(2)}} \text{grad}_y \bar{m}^2; \vec{y} = \vec{x}_1 - \vec{x}_2 \\ \longrightarrow \mathcal{H}_{(2)}^2 \frac{1}{4} (\dot{\vec{y}})^2 &= 4p_{c.m.}^2 + \bar{m}^2 = \mathcal{M}^2 = \text{constant}\end{aligned}\quad (92)$$

adopting the long range approximate nature of the harmonic oscillator relations – for the  $q\bar{q}$  – bond

$$\bar{m}^2 \sim_{|y| \rightarrow \infty} \left( \frac{1}{2} \Lambda \right)^2 y^2 [1 + O(M_q / |y|) + \dots] \quad (93)$$

In eq. 92 we have substituted the variable  $\vec{y}$  for  $z \rightarrow \vec{z}$ .  
A few remarks shall follow

- 1) The lessons from the  $q\bar{q}$  – bond are limited

The exclusive relative *distance*-dependence contained in the asymptotic term in eq. 93

$$\bar{m}^2 \sim \frac{1}{2} \Lambda^2 y^2 \quad (94)$$

generates genuine oscillatory modes for the  $q\bar{q}$  – bond , yet no multi-position dependent generalization can accomplish the same for the  $3q - (Nq -)$  bonds .

- 2) but not empty

For the  $q\bar{q}$  – bond it follows

$$\begin{aligned} \ddot{\vec{y}} &= - \left( \frac{\Lambda}{\mathcal{H}_{(2)}} \right) y \quad , \quad \Lambda [ - \Delta_\xi + \xi^2 ] = H_{(2)}^2 \\ \xi &= \frac{1}{2} (\Lambda)^{1/2} y \\ \longrightarrow \omega_{cl} &= \frac{\Lambda}{\mathcal{H}_{(2)}} \quad , \quad H_{(2)}^2(\{\nu\}) = 2 \Lambda \sum_{\alpha=1}^3 \nu_\alpha + 3 \Lambda \\ \nu_\beta &= 0, 1, \dots ; \beta = 1, 2, 3 ; \text{ oscillator occupation numbers} \end{aligned} \quad (95)$$

- 3) The color quantum number has vanished from the description

Vacuum - vacuum amplitudes of two colored local operators are not gauge invariant , provided local gauge invariance is not conserved 'completely' , to be defined including appropriate generalized boundary conditions , in QCD .

The wave functions on the other hand are not local .

- 4) Which are the dependences on quantum numbers like (light-) flavors and spin ?

The quantity  $\Lambda$  in eq. 94 , of dimension  $\text{mass}^2$  , is considered universal , i.e. does not depend on any other quantum numbers neither on quark masses, except on the occupation numbers of the oscillatory modes at hand .

### Extension to include the $N$ $q$ – bond

The key idea arose upon a discussion initiated by H. R. Dicke , concerning the feasibility and appropriateness to envisage a revision of the errors , as established by Loránd ( Roland v. ) Eötvös in 1918 , in his famous experiments in ref. [27-1918-1961] measuring the equality of inertial and gravitational mass with the help of rotating springs , to which test bodies are attached , while the springs are fixed to one point , say atop a rotating rod . It should be added that the springs must be elongating under the centrifugal force only in one longitudinal direction .

Dicke reports in ref. [27-1918-1961] that Eötvös , in his description of the experiments mentions an important obstacle to overcome , consisting in a precise separation of the mass of the test bodies from a combination of a part of the spring mass with them .

Hence the (my) conclusion from the above situation to the question envisaged was , that in presence of position dependent mass this mass and inertial mass were *not the same* .

This led to the Ansatz , as I followd in ref. [15-1980]

$$\begin{aligned}\mathcal{L}_N &= - \sum_{\alpha=1}^N \left[ m_\alpha^2 - Q_{\beta\gamma}^\alpha \vec{v}_\beta \cdot \vec{v}_\gamma \right]^{1/2} \\ \vec{v}_\alpha &= \dot{\vec{x}}_\alpha \\ m_\alpha &= m_\alpha [ \vec{z}_1, \dots, \vec{z}_{N-1} ] \quad , \quad \text{gravitational effective masses} \\ Q_{\beta\gamma}^\alpha &= Q_{\beta\gamma}^\alpha [ \vec{z}_1, \dots, \vec{z}_{N-1} ] \quad , \quad \text{inertial effective masses}\end{aligned}\tag{96}$$

valid in the c.m. system of the  $N$  quarks .

The external quark masses  $M_q$  , appropriate multipliers of the scalar densities  $\bar{q}q$  composing the mass term in the local (QCD) Lagrangian

$$- \mathcal{L}_m = \sum_{flavors} \frac{z_q}{z_M} M_q \bar{q}^c q^c \tag{97}$$

appear as constants in the gravitational mass

$$\begin{aligned}m_\alpha [ M_q , \underline{z} ] &= M_\alpha + m_\alpha [ M_q = 0 , \underline{z} ] \\ \underline{z} &= ( \vec{z}_1 , \dots , \vec{z}_{N=1} )\end{aligned}\tag{98}$$

whereas a consistent non-relativistic limit demands

$$\begin{aligned}[ Q_{\alpha\alpha}^\alpha ( M_q , \underline{z} ) ]^{1/2} &\xrightarrow{M_q \rightarrow \infty} M_\alpha + O [ m_\beta [ M_q = 0 , \underline{z} ] ] \\ Q_{\beta\gamma}^\alpha , \beta, \gamma \neq \alpha &\xrightarrow{M_q \rightarrow \infty} O [ m_\beta [ M_q = 0 , \underline{z} ] ]\end{aligned}\tag{99}$$

From eqs. 98 and 99 we recognize the problem of separation of mass and binding energy, as relevant, e.g., to the gravitational interaction of the whole  $N$ -quark system, appearing.

In the following  $m$ -,  $Q$ - are approximated by the corresponding quantities for  $M_q = 0$ , i.e., in the chiral limit with respect to all the quark flavors composing the  $N$   $q$ - bond.

Then in the harmonic long-range limit  $Q$ - is determined from  $m$ - through the relation

$$Q_{\beta\gamma}^{\alpha} = \frac{1}{K_N} (m_{\alpha})^2 \delta_{\beta\gamma}; \quad K_N : \text{constant} \quad (100)$$

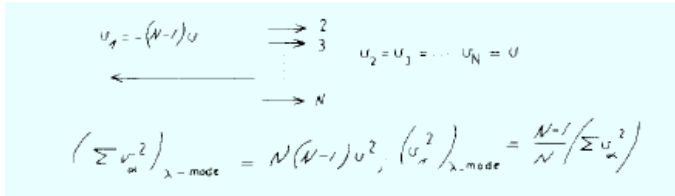
The kinetic term for the quark  $\alpha$  depends on all the velocities  $\vec{v}_{\beta}$  and the Lagrangean  $\mathcal{L}_N$  in eq. 96 takes the simplified form

$$\begin{aligned} \mathcal{L}_N &= -\bar{m} \left[ 1 - \sum_{\beta} (\vec{v}_{\beta})^2 \right]^{1/2}; \quad \bar{m} = \sum_{\alpha=1}^N m_{\alpha} = \bar{m} \left( \vec{x}_{\beta} - \vec{X} \right) \\ \vec{X} &= \frac{1}{N} \sum_{\alpha=1}^N \vec{x}_{\alpha} \rightarrow 0 \end{aligned} \quad (101)$$

The meaning of the constant  $K_N$  is the following : under the constraint  $\sum_{\alpha} \vec{v}_{\alpha} = 0$  the maximum any individual  $(\vec{v}_{\beta})^2$  can assume for given  $\sum_{\gamma} (\vec{v}_{\gamma})^2$  is for the so-called  $\lambda$ -mode, shown in figure 6 below.

An inequality for any individual square velocity follows

$$\begin{aligned} v_{\alpha} &= \left( (\vec{v}_{\alpha})^2 \right)^{1/2} \longrightarrow \\ v_{\alpha}^2 &\leq \frac{N-1}{N} \sum_{\gamma} v_{\gamma}^2 \leq \frac{N-1}{N} K_N c^2 \quad (c = 1) \end{aligned} \quad (102)$$



**Fig. 6 :**  $\lambda$  - mode for  $N$  quark bond  $\longleftrightarrow$

$\mathcal{L}_N$  in eq. 101 delimits its validity to physical values of  $v_{\alpha}^2$  for  $i$   
 $K_N = N / (N - 1)$

$$v_{\alpha}^2 \leq c^2 \longleftrightarrow K_N = \frac{N}{N - 1} \quad (103)$$

The equation for the conserved energy ( eq. 91 for  $\mathcal{L}_{(2)}$  ) for  $\mathcal{L}_N$  becomes

$$\begin{aligned}\mathcal{H}_N &= \vec{v}_\alpha \vec{p}_\alpha - \mathcal{L}_N \\ \vec{p}_\beta &= \mathcal{L}_N, \vec{v}_\beta = \frac{\overline{m}}{K_N} \left( [1 - \omega^2]^{-1/2} \right), \vec{v}_\beta = \frac{\mathcal{H}_N}{K_N} \vec{v}_\beta \\ \omega^2 &= \frac{1}{K_N} \sum_{\gamma=1}^N v_\gamma^2; K_N = \frac{N}{N-1}\end{aligned}\tag{104}$$

From eq. 104 we obtain in canonically conjugate oscillator variables

$$\begin{aligned}(\mathcal{H}_N)^2 &= \left[ K_N \sum_{\alpha=1}^N (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{p}_\beta=0} + \overline{m}^2 (x_\gamma - X) \right] \\ &= \left[ K_N \sum_{\alpha=1}^N (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{p}_\beta=0} + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^N (\vec{x}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{x}_\beta=0} \right]\end{aligned}\tag{105}$$

$\mathcal{H}_N$  is a constant of the motion by the relations displayed in eq. 104, but it is  $(\mathcal{H}_N)^2 \equiv \mathcal{M}_N^2$ , which becomes the genuinely canonical dynamic operator, or in the classical framework 'Hamiltonian function', in the genuinely relativistic situation.

The structure of  $\mathcal{M}_N^2$  is derived straightforwardly from eqs. 104 and 105

$$\begin{aligned}\mathcal{M}_N^2 &= \left[ K_N \sum_{\alpha=1}^N (\vec{p}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{p}_\beta=0} + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^N (\vec{x}_\alpha)^2 \Big|_{\sum_{\beta=1}^N \vec{x}_\beta=0} \right] \\ &= \left[ K_N \sum_{\alpha=1}^{N-1} (\vec{\pi}_\alpha)^2 + \frac{\Lambda^2}{K_N} \sum_{\alpha=1}^{N-1} (\vec{z}_\alpha)^2 \right] \\ (\vec{\pi}_\beta, \vec{z}_\beta) &: \text{barycentric coordinates defined in eq. 86; } K_N = \frac{N}{N-1}\end{aligned}\tag{106}$$

The mass-square spectrum according to  $\mathcal{M}_N^2$  in eq. 106 is given by

$$\begin{aligned}\mathcal{M}_N^2 \Big|_{\text{spectrum}} &= 2\Lambda \sum_{\alpha=1}^{3N-3} \nu_\alpha + 3\Lambda (3N-3) \\ \nu_\beta &= 0, 1, \dots; \beta = 1, 2, \dots, 3N-3\end{aligned}\tag{107}$$

Except for the zero point contribution, i. e. for the oscillation level splittings, it is universal, independent of N, proving the validity of the universal relation in eq. 87 ( $\Lambda_N / \Lambda = 1$ ).

Taking the  $\nu = \sum_{\eta=1}^6 \nu_{\eta} = 2$ ,  $P = +$  nonstrange baryon states, i. e. for  $N_{fl} = 2$ ,  $N = 3$  and counting in a reduced way, all corresponding oscillatory modes with positive parity, compatible with overall Bose symmetry, neglecting color, we obtain the following table of 21 states, not counting isospin and spin degrees of freedom separately

TABLE $\pi N$ -partial waves and associated number of baryon resonances with $\nu = 2, P = +$		
$\pi N$ partial wave	Number of states with $\nu = 2, P = +$	Candidates
P11 $N(\frac{1}{2}^+)$	4	2
P13 $N(\frac{3}{2}^+)$	5	1-2
F15 $N(\frac{3}{2}^+)$	3	2
F17 $N(\frac{3}{2}^+)$	1	1
P31 $\Delta(\frac{1}{2}^+)$	2	1-2
P33 $\Delta(\frac{3}{2}^+)$	3	1
F35 $\Delta(\frac{3}{2}^+)$	2	1
F37 $\Delta(\frac{3}{2}^+)$	1	1
	<b>21</b>	<b>10-12</b>

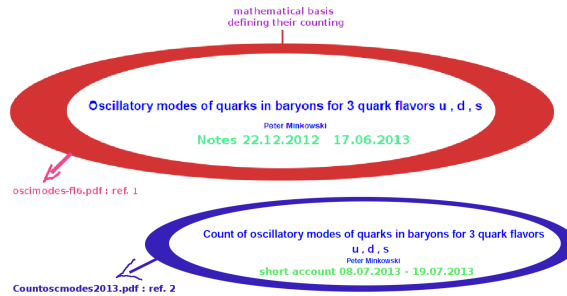
**Fig. 7 : Nonstrange baryons with  $\nu = 2, P = +$  in 1980  $\longleftrightarrow$**

The candidates were collected in ref. [15-1980] from the PDG tables valid in 1980. At least then almost 50 % of the resonances so characterized were missing, using this way of counting.

## 5 – First results from counting oscillatory modes in u,d,s flavored baryons ( [28-2013] )

Perspectives : A. How to count oscillatory modes of quarks in baryons for 3 quark flavors u, d, s ;  $t \geq 0$ .

In the 2 figure captions below refs. 1, 2  $\rightarrow$  29, 30.



ref. 29 : [29-2013] ; ref. 30 : [30-2013]

B. Tuning to harmonic numbers of oscimodes of baryons



## 1 - Introduction'

The perspectives illustrated in Figs. 1 and 2 are meant to apply to present and future derivations . In this sense refs. 29 and 30 – [29-2013] and [30-2013] – refer to recent results . The comparison of hadron yields measured at RHIC and LHC with a noninteracting hadron resonance gas necessitates the counting of these resonances , which is not obvious. This is illustrated in Fig. 1 of ref. 3 [31-2010] , reproduced as Fig. 3 below . Thus the problem of identifying 'oscillatory modes of light quark flavors in baryons' , presented in ref. 15 – [15-1980] – took center-stage . Having read around the year 1976 a paper on relativistic oscillator solutions to the Dirac equation , only very recently I could reconstruct its quotation, which becomes ref. 32 – [32-1975].<sup>3</sup>

Here I focus on the cornerstones , which allow to count these oscillatory modes , as outlined in extenso in refs. 29 and 30 , op.cit. , beginning with the classification of the representations of  $S_3$  – the permutation group of the three quarks in configuration space – as they arise through the induced representation from the associated wave functions in the subsequent sections

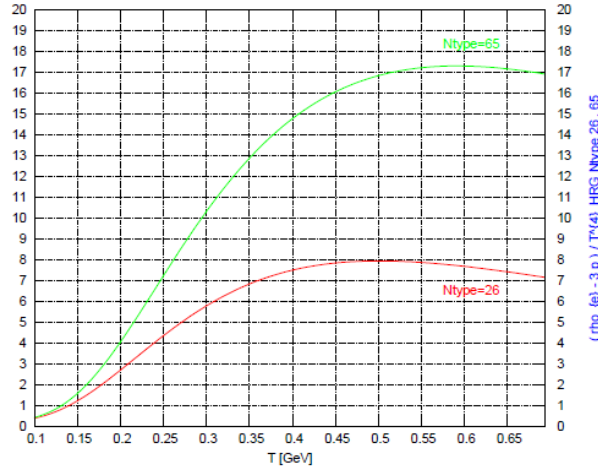


Fig. 3 : The thermal quantity representing the trace anomaly – dscale  $(T) = (\rho_e - 3p) / T^4$  is shown for  $0.1 \text{ GeV} \leq T \leq 0.7 \text{ GeV}$  . The comparison of HRG collections Ntype=65 and 26 shows the sensitive temperature regions beyond temperatures where chemical freeze out takes place –  $T \sim 150 - 170 \text{ MeV}$  for Pb - Pb collisions at SPS and Au - Au collisions at RHIC.

<sup>3</sup> I am indebted to Christoph Greub for reminding me of the first article by C. L. Critchfield – [32-1975] – on scalar potentials .



## 2 - Factoring out the *approximate* symmetry group in spin-flavor space as well as overall color

It is inherent to the path followed to point out the historical formulation of the 'bootstrap hypothesis' underlying and restricting the full set of S-matrix elements pertaining to strongly interacting hadrons , due to G. F. Chew e.g. in ref. 33 – [33-1962] . A good textbook reference is ref. 34 – [34-1968] .

In this context all resonances observed or hypothetically to be observed in scattering of stable hadrons , irrespective of their width , are considered to be hadrons . A central notion within the 'bootstrap'-framework is the density of hadrons and its limiting behaviour for large mass-square

$$\varrho_n ( m^2 ) = \frac{\partial \mathbb{N} ( m^2 )}{\partial m^2} \quad (108)$$

4

In the density with respect to mass square  $\varrho_n$  defined in eq. 108 the density per phase space of an isolated state of momentum  $\vec{p}$  is not included

$$d\Phi = V ( 2\pi )^{-3} d^3 p ; V : \text{space volume} \quad (109)$$

The parametrization defined in eq. 108 is directly applicable to counting resonances in the particle listings of the PDG [20-2012] . To this end a binning in mass square is to be chosen and a histogram of resonance counts per bin yields the so approximated density function  $\varrho_n$  .

In 1965 Rolf Hagedorn ( 1919 - 2003 ) wrote an elaborate paper – [35-1965] – centered around the hypothesis of the limiting behaviour of the quantity  $\varrho_n$  for  $m \rightarrow \infty$  as a solution to the bootstrap conditions

$$\varrho_n ( m^2 ) \sim \left( \frac{m^2}{m_0^2} \right)^a \exp ( m / T_0 ) \text{ for } m \rightarrow \infty \quad (110)$$

$m_0 , a , T_0$  : characteristic parameters

In 1968 Gabriele Veneziano – [36-1968] – arrived at a solution to the 'bootstrap' idea starting from the decay amplitude for the process

$$\omega \rightarrow \pi^+ \pi^- \pi^0 \quad (111)$$

reproducing Regge poles in all two particle channels upon suitable extrapolations in their respective momenta , with generalizations to multiparticle amplitudes called 'dual' . The structure underlying the totality of dual amplitudes is the quantum mechanical motion of a one dimensional open (super)string with constant string tension  $T = 1 / ( 2\pi \alpha' )$  , whose

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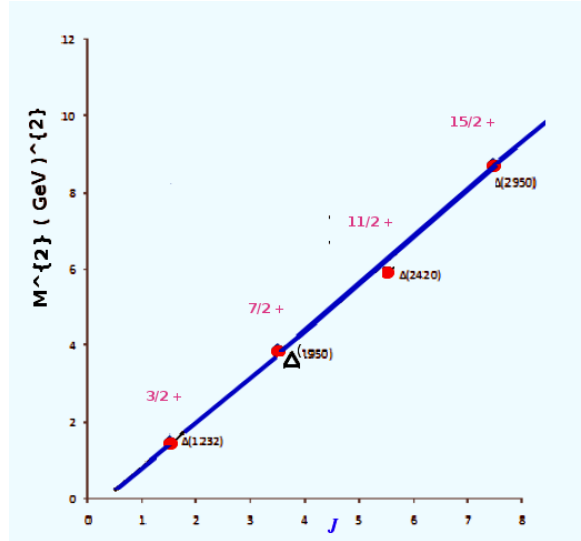
<sup>4</sup> We use the symbol  $\mathbb{N}$  for numbering integers except the number of valence quarks forming a baryon , denoted  $N$  .

harmonic vibrations generate linear  $\left( J \text{ versus } \alpha' m^2 \right)$  bosonic and fermionic Regge trajectories . A review may be found in ref. 37 , [37-1982]. In the domain of numbers the counting of *partitions* , i.e. the power of the set of nonnegative integers  $n_1, n_2, \dots, n_\infty$  with  $n_k = 0, 1, \dots, \infty \forall k$

$$\wp(N) = \{ n_1, n_2 \dots n_\infty \mid \sum_{k=1}^{\infty} k n_k = N \} \quad (112)$$

$$n_1, n_2 \dots n_\infty = 0, 1, \dots$$

determines by its asymptotic behaviour for  $N \rightarrow \infty$  that indeed the free superstring possesses a similar growth as in eq. 110 – modulo multiplicative logarithmic factors in the exponent – such that a maximal temperature exists in accordance with Hagedorn's hypothesis . This is worked out in ref. 38 – [38-1986] . In a specific case of open superstrings on noncommutative space eq. 110 is reproduced with  $a = -\frac{9}{4}$  in ref. 39 – [39-2000] .  $\wp(N)$  in eq. 112 is treated in ref. 40 – [40-1970] , pp. 822ff.



**Fig. 4 : Regge trajectory of  $\Delta$  baryons comprising  $2 J^P = (3, 7, 11, 15)^+$**

## 6 - Concluding remarks , outlook

- 1) A different view on resonances from QCD sum rules and condensates, also relating to supersymmetric QCD by Adi Armoni and Mikhail Shifman can be found in ref. 41 – [41-2003] .
- 2) The  $\left( 70 \times \vec{L} = 1 \right)^-$  negative parity u , d , s baryon multiplet with  $N = 1$  is well described in the current PDG review – [20-2012] – 'Quark Model' by C. Amsler,

T. De Grand and B. Krusche in ref. 19 – [19-2011] . Extended other baryon- and meson multiplets including heavy quark flavors c , b are assigned quark and antiquark configurations as well . A pioneering paper discussing u, d, s flavored  $\bar{q}' q$  mesons is due to George Zweig [42-1968] .

3) The circular pair-mode oscillator basis

is presented in detail in ref. 29 – [29-2013] . Here space unfortunately does not allow me to cover this topic , instrumental to establish the counting of oscillatory modes in u, d, s - baryons .

4) The  $\mathbb{N}$  – density of baryon states per mass-square

defined in eq. 108

$$\varrho_n ( m^2 ) = \frac{\partial \mathbb{N} ( m^2 )}{\partial m^2} \quad (113)$$

behaves for large  $\mathbb{N}$  like  $\mathbb{N}^u$  ;  $u = 5$  . This is enough to establish that string modes and oscillator modes presented are *inequivalent* .

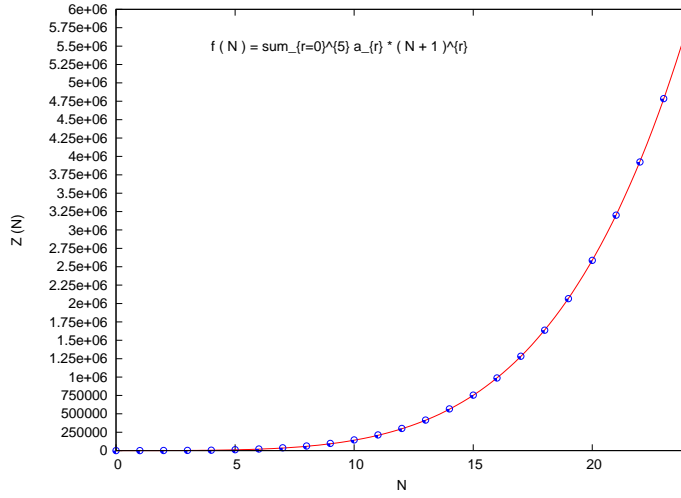


Fig. 5 :  $\# ( \mathbb{N} )$

Counting all u, d, s baryon states in the PDG – [20-2012] with total spin and isospin multiplicities accounted for, 1274 are obtained . This was done omitting a few very doubtful resonances .

The count of states with  $N \leq N^*$  gives

$N^*$	#
0	56
1	266
2	1310
3	4090

with  $N^* \sim 3$  being a fair estimate of resonances up to 2.5 - 3 GeV .

#### 5) outlook

I hope that at the high energy frontier , despite the odds looking unfavorable , exploiting the increased production cross sections of hitherto unobserved resonances , the art of resonance spectroscopy – even of low energy resonances – can witness a new frontier .

### Appendix 1: The spin (10) product representations $(16 \oplus \overline{16}) \otimes (16 \oplus \overline{16})$

We follow the spin (10) decomposition discussed in section 2-1 ( eq. 27 repeated below )

$$\text{spin}(10) \rightarrow \text{SU}5 \times \text{U}1_{J_5} \quad (114)$$

Further let us denote representations of spin (10) as opposed to those pertaining to SU5 and associated  $J_5$  quantum number by

$$\text{spin}(10) : [dim] ; \text{SU}5 \times \text{U}1_{J_5} : \{dim\}_{J_5} \quad (115)$$

Thus eq. 139 translates to

$$\begin{aligned} [16] &= \{1\}_{+5} + \{10\}_{+1} + \{\overline{5}\}_{-3} \\ [\overline{16}] &= \{1\}_{-5} + \{\overline{10}\}_{-1} + \{5\}_{+3} \end{aligned} \quad (116)$$

In turn SU5 representations shall be decomposed along the standard model gauge group  $\text{SU}3_c \otimes \text{SU}2_L \otimes \text{U}1_{\mathcal{Y}}$  , where  $\mathcal{Y}$  denotes the electroweak hypercharge (with a factor  $\frac{1}{2}$  included )

$$\mathcal{Y} = Q_{e.m.} / e - I_{3L} \quad (117)$$

$$\{dim\} \rightarrow \sum ](dim\text{SU}3_c , dim\text{SU}2_L)_{\mathcal{Y}}[ \quad (118)$$

The brackets on the right hand side of eq. 118 are reversed in order not to confuse spin (10) - and standard model representations.

Then the base  $16 \oplus \overline{16}$  decompose to

$$[16] \rightarrow \begin{cases} \{1\}_{+5} \rightarrow \{ \} (1, 1)_0 \{ \}_{+5} \\ \{10\}_{+1} \rightarrow \left\{ \begin{array}{l} \left[ \begin{array}{l} (3, 2)_{+\frac{1}{6}} \\ (\overline{3}, 1)_{-\frac{2}{3}} \end{array} \right] \left[ \begin{array}{l} + \\ + \end{array} \right] \\ (1, 1)_{+1} \left[ \begin{array}{l} + \\ + \end{array} \right] \end{array} \right\}_{+1} \\ \{\overline{5}\}_{-3} \rightarrow \left\{ \begin{array}{l} \left[ \begin{array}{l} (\overline{3}, 1)_{+\frac{1}{3}} \\ (1, 2)_{-\frac{1}{2}} \end{array} \right] \left[ \begin{array}{l} + \\ + \end{array} \right] \end{array} \right\}_{-3} \end{cases} \quad (119)$$

The product representations  $(16 \oplus \overline{16}) \otimes (16 \oplus \overline{16})$  generate *all* SO (10) antisymmetric tensor ones, of which we encountered the fivefold antisymmetric in section 2-1 (eq. 144).

To elaborate we specify the n-fold antisymmetric tensors obtained from the 10-representation of SO (10)

$$\begin{aligned} [t_0] &\sim 1 \\ [t_1]^A &\sim z^A ; A = 1, 2, \dots, 10 \leftrightarrow [t_1] = \{\overline{5}\}_2 \oplus \{5\}_{-2} \\ [t_2]^{[A_1 A_2]} &\sim \frac{1}{2} \left( z_1^{A_1} z_2^{A_2} - z_1^{A_2} z_2^{A_1} \right) \\ &\dots \\ [t_n]^{[A_1 A_2 \dots A_n]} &\sim \frac{1}{n!} \sum sgn \left( \begin{array}{ccc} 1 & \dots & n \\ \pi_1 & \dots & \pi_n \end{array} \right) z_1^{A_{\pi_1}} z_2^{A_{\pi_2}} \dots z_n^{A_{\pi_n}} \\ n &\leq 10 \end{aligned} \quad (120)$$

The quantities  $[t_n]$  defined in eq. 120 form irreducible real representations of SO (10) except for  $n = 5$ , which is composed of the relatively complex *irreducible* representations  $126$  and  $\overline{126}$  (eq. 144).

The tenfold antisymmetric invariant corresponds to  $[t_{n=10}]$ . The product of two full Clifford algebras pertaining to spin (10) contains all  $[t_n]$ ;  $n = 0 \dots 10$  representations exactly once.

Treating the  $n = 5$  tensor as one representation – it is reducible only over  $\mathbb{C}$  – the dimensions of the  $[t_n]$  representations follow Pascal's triangle (Fig. 3 page C7) of binomial coefficients for  $N = 10$ , whereby  $n$  even and odd shall be distinguished

$$\begin{array}{cccccc} [t_0] & [t_2] & [t_4] & [t_6] & [t_8] & [t_{10}] \\ & [t_1] & [t_3] & [t_5] & [t_7] & [t_9] \\ 1 & 45 & 210 & 210 & 45 & 1 \\ & 10 & 120 & 252 & 120 & 10 \end{array} \quad (121)$$

This corresponds to the following products of  $16 + \overline{16}$

	$[16]$		$\overline{[16]}$	
$[16]$	$s : \begin{matrix} [10] + \\ [126] \end{matrix}, a : [120]$		$[1] + [45] + [210]$	(122)
$\overline{[16]}$	$[1] + [45] + [210]$		$s : \begin{matrix} [10] + \\ \overline{[126]} \end{matrix}, a : [120]$	

The correspondence of product representations of the  $16 + \overline{16} = 32$  associative Clifford algebra with the sum of antisymmetric tensor ones follows from the completeness of all products of  $\gamma$  matrices forming the spin (10) algebra i.e. are of dimension

$$(32)^2 = (2^5)^2 = 2^{10} \quad (123)$$

We proceed to reduce the  $[16] \otimes [16]$  product with respect to  $J_5$ , SU5 and  $SU3_c \times SU2_L \times U1_Y$ .

The individual products are ( $s(a)$  : (a)symmetric )

	$\{1\}_5$	$\{10\}_1$	$\{\overline{5}\}_{-3}$
$\{1\}_5$	$\{1\}_{10s}$	$\{10\}_6$	$\{\overline{5}\}_2$
$\{10\}_1$	$\{10\}_6 \left( \begin{matrix} \{\overline{5}\}_2 + \\ \{\overline{50}\}_2 \end{matrix} \right)_s$	$(\{\overline{45}\}_2)_a$	$\left( \begin{matrix} \{5\}_{-2} + \\ \{45\}_{-2} \end{matrix} \right)$
$\{\overline{5}\}_{-3}$	$\{\overline{5}\}_2$	$\left( \begin{matrix} \{5\}_{-2} + \\ \{45\}_{-2} \end{matrix} \right)$	$(\{\overline{15}\}_{-6})_s (\{\overline{10}\}_{-6})_a$

(124)

We proceed to decompose the diagonal  $\{SU5\}_{J_5}$  representations (eq. 119)

$$(\{10\}_1 \otimes \{10\}_{-1})_s = \{\overline{5}\}_2 + \{\overline{50}\}_2 \quad \downarrow$$

$s$	$\left[ \begin{matrix} (3, 2)_{+\frac{1}{6}} \\ (3, 1)_{-\frac{2}{3}} \end{matrix} \right]_{+1}$	$\left[ \begin{matrix} (3, 2)_{+\frac{1}{6}} \\ (3, 1)_{-\frac{2}{3}} \end{matrix} \right]_{+1}$	$\left[ \begin{matrix} (3, 2)_{+\frac{1}{6}} \\ (3, 1)_{-\frac{2}{3}} \end{matrix} \right]_{+1}$	(125)
$\left[ \begin{matrix} (3, 2)_{+\frac{1}{6}} \\ (3, 1)_{-\frac{2}{3}} \end{matrix} \right]_{+1}$	$\left( \left[ \begin{matrix} (6, 3)_{+\frac{1}{3}} \\ (\overline{3}, 1)_{+\frac{1}{3}} \end{matrix} \right]_2 \right)^+$	$\left( \left[ \begin{matrix} (8, 2)_{-\frac{1}{2}} \\ (1, 2)_{-\frac{1}{2}} \end{matrix} \right]_2 \right)^+$	$\left[ \begin{matrix} (3, 2)_{+\frac{1}{6}} \\ (3, 1)_{-\frac{2}{3}} \end{matrix} \right]_2$	
$\left[ \begin{matrix} (\overline{3}, 1)_{-\frac{2}{3}} \\ (1, 1)_{+1} \end{matrix} \right]_{+1}$	$\left[ \begin{matrix} (\overline{6}, 1)_{-\frac{4}{3}} \\ (\overline{3}, 1)_{+\frac{1}{3}} \end{matrix} \right]_2$	$\left[ \begin{matrix} (\overline{6}, 1)_{-\frac{4}{3}} \\ (\overline{3}, 1)_{+\frac{1}{3}} \end{matrix} \right]_2$	$\left[ \begin{matrix} (\overline{6}, 1)_{-\frac{4}{3}} \\ (\overline{3}, 1)_{+\frac{1}{3}} \end{matrix} \right]_2$	
$\left[ \begin{matrix} (1, 1)_{+1} \\ (1, 1)_{+1} \end{matrix} \right]_{+1}$	$\left[ \begin{matrix} (1, 1)_{+1} \\ (1, 1)_{+1} \end{matrix} \right]_2$	$\left[ \begin{matrix} (1, 1)_{+1} \\ (1, 1)_{+1} \end{matrix} \right]_2$	$\left[ \begin{matrix} (1, 1)_{+1} \\ (1, 1)_{+1} \end{matrix} \right]_2$	

$$\begin{aligned}
& \left( \{\bar{5}\}_{-3} \otimes \{\bar{5}\}_{-3} \right)_s = \{\bar{15}\}_{-6} \quad \downarrow \\
& \begin{array}{c|c} s & \left[ \begin{array}{c} (\bar{3}, 1)_{+\frac{1}{3}} \left[ \begin{array}{c} -3 \end{array} \right] (1, 2)_{-\frac{1}{2}} \left[ \begin{array}{c} -3 \end{array} \right] \\ \hline \left[ \begin{array}{c} (\bar{3}, 1)_{+\frac{1}{3}} \left[ \begin{array}{c} -3 \end{array} \right] \end{array} \right] \left[ \begin{array}{c} (\bar{6}, 1)_{+\frac{2}{3}} \left[ \begin{array}{c} -6 \end{array} \right] \end{array} \right] (\bar{3}, 2)_{-\frac{1}{6}} \left[ \begin{array}{c} -6 \end{array} \right] \\ \left[ \begin{array}{c} (1, 2)_{-\frac{1}{2}} \left[ \begin{array}{c} -3 \end{array} \right] \end{array} \right] \left[ \begin{array}{c} (1, 3)_{-1} \left[ \begin{array}{c} -6 \end{array} \right] \end{array} \right] \end{array} \right. \\
\end{array} \quad (126) \\
& \quad \downarrow \\
& \text{complex e.w. triplet coupling to} \\
& \quad \frac{1}{2} \left( \nu_{\bar{F}}^* \right)^\alpha \left( \nu_{\bar{G}}^* \right)_\alpha
\end{aligned}$$

Next we assemble the (anti)symmetric products  $([16] \otimes [16])_s = [10] \oplus [126]$  and  $([16] \otimes [16])_a = [120]$  with respect to  $SU5 \otimes U1_{J_5}$  using eq. 124

$$\begin{aligned}
& ([16] \otimes [16])_s = [10] \oplus [126] \quad \downarrow \\
& = \left\{ \begin{array}{c} \left[ \begin{array}{c} \{5\}_{-2} + \\ \{\bar{5}\}_2 \end{array} \right] \\ \oplus \left[ \begin{array}{c} \{1\}_{10} + \{\bar{5}\}_{II} + \{10\}_6 + \{\bar{15}\}_{-6} \\ + \{45\}_{-2} + \{\bar{50}\}_2 \end{array} \right] \end{array} \right\} \quad (127) \\
& ([16] \otimes [16])_a = [120] \quad \downarrow \\
& = \left[ \begin{array}{c} \{5\}_{-2} + \{\bar{5}\}_2 \\ + \{10\}_6 + \{\bar{10}\}_{-6} \\ + \{45\}_{-2} + \{\bar{45}\}_2 \end{array} \right]
\end{aligned}$$

The roman indices  $_{I,II}$  in eq. 127 indicate that appropriate linear combinations of the *two*  $\{\bar{5}\}_2$  representations form parts of  $[10]$  and  $[126]$  respectively .

It remains to decompose the  $SU5 \otimes U1_{J_5}$  representations in eq. 127 with respect to  $SU3_c \times SU2_L \times U1_y$  . We do this associating according to

the product representations as they appear in eq. 127

$$\begin{array}{c|c}
\begin{array}{l} [10] \quad [120] \quad \{5\}_{-2} \\ [10] \quad [126] \quad [120] \quad \{\overline{5}\}_{+2} \end{array} & \begin{array}{l} \left[ (3,1)_{-\frac{1}{3}} \left[ \begin{smallmatrix} + \\ +3 \end{smallmatrix} \right] (1, \overline{2})_{+\frac{1}{2}} \left[ \begin{smallmatrix} + \\ +3 \end{smallmatrix} \right] \right. \\ \left. \left[ (\overline{3},1)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ -3 \end{smallmatrix} \right] (1,2)_{-\frac{1}{2}} \left[ \begin{smallmatrix} + \\ +3 \end{smallmatrix} \right] \right] \right. \end{array} \\
\hline
\begin{array}{l} [126] \quad \{1\}_{+10} \\ [126] \quad [120] \quad \{10\}_{+6} \\ [120] \quad \{\overline{10}\}_{-6} \end{array} & \begin{array}{l} \left[ (1,1)_0 \left[ \begin{smallmatrix} + \\ +10 \end{smallmatrix} \right] \right. \\ \left[ (3,2)_{-\frac{1}{6}} \left[ \begin{smallmatrix} + \\ +6 \end{smallmatrix} \right] (\overline{3},1)_{-\frac{2}{3}} \left[ \begin{smallmatrix} + \\ +6 \end{smallmatrix} \right] (1,1)_{+1} \left[ \begin{smallmatrix} + \\ +6 \end{smallmatrix} \right] \right. \\ \left. \left[ (\overline{3}, \overline{2})_{+\frac{1}{6}} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] (3,1)_{+\frac{2}{3}} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] (1,1)_{-1} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] \right] \right. \end{array} \\
\end{array} \tag{128}$$

$$\begin{array}{c|c}
\begin{array}{l} [126] \quad \{\overline{15}\}_{-6} \\ [126] \quad [120] \quad \{45\}_{-2} \\ [120] \quad \{\overline{45}\}_{+2} \end{array} & \begin{array}{l} \left[ (\overline{6},1)_{+\frac{2}{3}} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] (\overline{3},2)_{-\frac{1}{6}} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] (1,3)_{-1} \left[ \begin{smallmatrix} + \\ -6 \end{smallmatrix} \right] \right. \\ \begin{array}{c} c.c. \updownarrow \\ \left[ \begin{array}{l} \left( \left[ (6,1)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) + \left( \left[ (8,2)_{-\frac{1}{2}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) + \right. \\ \left. \left[ (\overline{3},3)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) + \left( \left[ (1,2)_{-\frac{1}{2}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) + \right. \\ \left. \left[ (3,2)_{+\frac{7}{6}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right] (3,1)_{-\frac{4}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] (\overline{3},1)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right] \end{array} \right. \end{array} \\ \hline \begin{array}{l} [126] \quad \{\overline{50}\}_{+2} \end{array} & \begin{array}{l} \left[ \begin{array}{l} \left( \left[ (6,3)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) + \left[ (8,2)_{-\frac{1}{2}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right] (3,2)_{+\frac{7}{6}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right. \\ \left. \left. + \left[ (\overline{6},1)_{-\frac{4}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right] (1,1)_{+2} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right] \right. \end{array} \right. \end{array} \\
\end{array} \tag{129}$$

$$(\{10\}_{-1} \otimes \{10\}_{-1})_s = \{\overline{45}\}_2 \quad \downarrow$$

$$\begin{array}{c|c}
a & \left[ (3,2)_{+\frac{1}{6}} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] (\overline{3},1)_{-\frac{2}{3}} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] (1,1)_{+1} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] \right. \\
\hline
\left[ (3,2)_{+\frac{1}{6}} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] \right. & \left( \left[ (6,1)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) \left( \left[ (8,2)_{-\frac{1}{2}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right) \right] (3,2)_{+\frac{7}{6}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \\
\left[ (\overline{3},1)_{-\frac{2}{3}} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] \right. & \left[ (3,1)_{-\frac{4}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] (\overline{3},1)_{+\frac{1}{3}} \left[ \begin{smallmatrix} + \\ 2 \end{smallmatrix} \right] \right. \\
\left[ (1,1)_{+1} \left[ \begin{smallmatrix} + \\ +1 \end{smallmatrix} \right] \right. & \text{---} \\
\end{array} \tag{130}$$



I end the collection of representation decompositions with the adjoint [45] representation of SO (10)

$$\begin{array}{c}
 ([10] \otimes [10])_a = [45] \quad \downarrow \\
 \begin{array}{c|cc}
 a & \{5\}_{-2} & \{\bar{5}\}_{+2} \\
 \hline
 \{5\}_{-2} & \{10\}_{-4} & \left\{ \begin{array}{l} \{1\}_0 \leftrightarrow J_5 \\ \{24\}_0 \leftrightarrow \text{adjoint SU5} \end{array} \right\} \\
 \{\bar{5}\}_{+2} & & \{\bar{10}\}_{+4}
 \end{array}
 \end{array} \quad (131)$$

It should be noted that despite coinciding dimensions the following entities are most distinct

$$\begin{aligned}
 [10] &\neq \{10\}_{-4} , \quad \{10\}_6 \\
 [45] &\neq \{45\}_{-2} ; \dots
 \end{aligned} \quad (132)$$

### Some conclusions from sections 1-1 and 1-2

- C1) The oscillation phenomena indicate clearly , that a *genuinely chiral* extension of B - L to a conserved, global symmetry, generating a *continuous* U1 - group of transformations, is not involved.
- C2) On the other hand the binary code of a ( minimally) supposed unifying gauge group SO or spin (10) could, if B - L is *not* gauged, equivalently generate a global symmetry of the vectorlike nature. The latter however would allow neutrino mass through the ( electroweak doublet-singlet ) pairing

$$-\mathcal{L}_M = \mu_{FG} \mathcal{N}_{\dot{\gamma}}^F \nu^{\dot{\gamma}G} + h.c. ; \quad F, G = 1, 2, 3 \text{ family} \quad (133)$$

without symmetry restrictions on the mass matrix  $\mu_{FG}$  in eq. 133.

- C3) Then however the question arises, why the mass matrix  $\mu$  , involving the scalar doublet(s) within the electroweak gauge group, also generating masses of charged spin  $\frac{1}{2}$  fermions, gives rise to very small physical neutrino masses. Thus we follow the *hypothesis* that SO (10) *is* gauged and that it is the *large* mass scale of the gauge boson associated with B - L in particular, which distinguishes neutrino flavors [3-1975] [8s2-1975], [s6-1976] .

**2-1+ The Majorana logic [s7-1994] and mass from mixing –  
setting within the 'tilt to the left' or 'seesaw' of type I ( ... )  
characterized by  $\mathcal{N}_F$**

Within the subgroup decompositions of SO (10) the 'tilt to the left' does not appear obvious

$$\begin{array}{ccc}
 & \text{spin (10)} & \\
 \swarrow & & \searrow \\
 \text{spin (6)} \equiv \text{SU4} & \times & \text{spin (4)} \equiv \text{SU2}_L \times \text{SU2}_R \\
 \text{lepton number} & & \\
 \text{as 4th color [s8-1974]} & & \\
 \downarrow & & \downarrow \\
 \text{SU3}_c \times \text{U1}_{B-L} & \times & \text{SU2}_L \times \text{U1}_{I_{3R}} \\
 \swarrow & & \swarrow \\
 & \text{SU3}_c \times \text{U1}_{Q_{e.m.}} & \\
 & Q_{e.m.} / e = I_{3L} + I_{3R} + \frac{1}{2} (B - L) & 
 \end{array} \tag{134}$$

In eq. 134 the conserved charge-like gauges are marked especially. The large scale breaking of *gauged* B - L or 'tilt to the left' was not assumed essential in refs. [3-1975] - [s6-1976] and brings about a definite 'mass from mixing' scenario [s9-1977] , [s10-1994] , [s11-1979] to which we turn below.

**The Majorana logic characterized by  $\mathcal{N}_F$  +**

Here we consider the alternative subgroup decomposition

$$\text{spin (10)} \rightarrow \text{SU5} \times \text{U1}_{J_5} \tag{135}$$

Among the 3 generators of spin (10) commuting with  $\text{SU3}_c$  ,  $I_{3L}$  ,  $I_{3R}$  and  $B - L$  and forming part of the Cartan subalgebra of spin (10) there is one combination, denoted  $J_5$  in eq. 135, commuting with its subgroup SU5 .

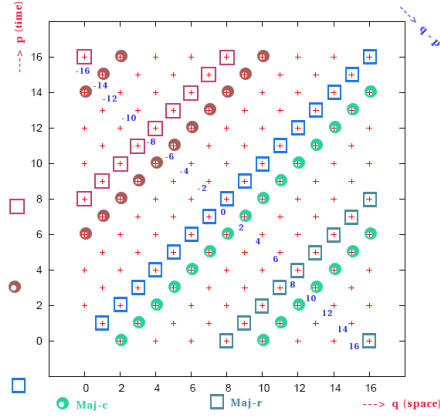
The 16 representation in the left-chiral basis displays the charges pertinent to  $J_5$  normalized to integer values *modulo an overall sign*, as in the discussion of genuinely chiral U1-charges in eq. 163 – but referring to N = 16

While the Majorana logic indeed opens a 'path' to trace the origin of the 'tilt to the left' , the origin of three families is unexplained at this stage.

The associative Clifford algebras  $\{ \Gamma_{p,q} ; \mathbb{C} \} \supset \{ \Gamma_{\tilde{p},\tilde{q}} ; \mathbb{R} \}$  are constructed in sections 4-1a  $\rightarrow$  4-1c, 4-2 and Appendices A, B forming complementary material to the present outline .

p , q denote time like ( p ) and spacelike ( q ) dimensions of space-time .

Fig. B1 shows the repartition of real ( Maj-r ) and complex ( Maj-c ) character of irreducible *associative* , real ( Majorana ) Clifford algebras



**Fig B1 : The complex and real Majorana representations**  
**MajCR** (  $p$  ,  $q$  )  $\longleftrightarrow$

with their characteristic mod 8 property relative to  $q - p$  [s1-1982]. These representations form the roots of the 'Majorana logic' discussed below .

$$\begin{aligned}
 (f)^{\dot{\gamma}} &= \left( \begin{array}{cccc|cccc} u^1 & u^2 & u^3 & \nu & \mathcal{N} & \hat{u}^3 & \hat{u}^2 & \hat{u}^1 \\ d^1 & d^2 & d^3 & e^- & e^+ & \hat{d}^3 & \hat{d}^2 & \hat{d}^1 \end{array} \right)^{\dot{\gamma} \rightarrow L} \\
 J_5 &\rightarrow \left( \begin{array}{cccc|cccc} 1 & 1 & 1 & -3 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -3 & 1 & -3 & -3 & -3 \end{array} \right)
 \end{aligned} \tag{136}$$

The assignment of  $J_5$  - charges in eq. 136 follows from the fermionic oscillator representation of the spin (2n) associated  $\Gamma$  algebra through n such oscillators and the associated embedding  $\text{spin}(10) \supset \text{SU5}$  [s13-1974] for  $n = 5$  here [s14-1980]

$$\begin{aligned}
 \{a_s, a_t^\dagger\} &= \delta_{st} ; s, t = 1, 2 \dots, n ; \{a_s, a_t\} = 0 = \{a_s^\dagger, a_t^\dagger\} \rightarrow \\
 J_n &= \sum_{s=1}^n \left( \begin{array}{cc} a_s^\dagger & a_s \\ -a_s & a_s^\dagger \end{array} \right) = 2\hat{n} - n \mathbb{I}_{2^n \times 2^n} ; \hat{n} = \sum_{s=1}^n a_s^\dagger a_s
 \end{aligned} \tag{137}$$

The eigenvalues (X) and multiplicities (#) of  $J_n$

$$\begin{array}{c|cccccc}
 (X) & n & n-2 & n-4 & \dots & -n+2 & -n \\
 \hline
 (\#) & \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \dots & \binom{n}{n-1} & \binom{n}{n}
 \end{array} \tag{138}$$

The orthogonal series for n even  $\leftrightarrow$  real ( spin (8) , spin(12)  $\dots$  ) has another decomposition within the associated  $\Gamma$  algebra , than the one with

$n$  odd  $\leftrightarrow$  complex ( spin (10) , spin (14)  $\dots$  ) . We give here the explicit numbers according to eq. 138 for  $n = 5$  , i.e. spin (10)

$(X)$	5	3	1	-1	-3	-5	
$(\#)$	$\begin{pmatrix} 5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 5 \end{pmatrix}$	(139)
SU5	$\{1\}$	$\{5\}$	$\{10\}$	$\{\overline{10}\}$	$\{\overline{5}\}$	$\{\overline{1}\}$	

The subset of states in blue in eq. 139  $(X) = \{5, 1, -3\}$  forms the 16 representation of spin 10, while those in red  $(X) = \{3, -1, -5\}$  the complex conjugate  $\overline{16}$  .

This opens the 'path' of linking the 'tilt to the left' with a substructure based on the primary in strength breakdown of the local gauged charginelike symmetry associated with

$$J_5 = -4 I_{3R} + 3 (B - L) \quad (140)$$

$J_5$  as defined through integer eigenvalues  $(X)$  given in eqs. 136 and 139 is normalized differently from the other Cartan subalgebra charges  $I_{3L}, I_{3R}, B - L$

$$\begin{aligned} |Q_C|^2 &= \sum_{\{16\}} (Q_C(f))^2, \quad |I_{3L}|^2 = 2, \quad |I_{3R}|^2 = 2 \\ |B - L|^2 &= \frac{16}{3}, \quad |J_5|^2 = 80 \end{aligned} \quad (141)$$

The consequence as far as neutrino-mass and mixing is concerned follows from identifying the  $J_5$  direction with a major axis of primary spontaneous gauge-symmetry breaking , bringing about the 'tilt to the left' from eq. 133

$$\begin{aligned} \mathcal{H}_M &= \mu_{FG} \mathcal{N}_{\dot{\gamma}}^F \nu^{\dot{\gamma}G} + h.c. + \mathcal{H}_M \\ \mathcal{H}_M &= \frac{1}{2} M_{FG} \mathcal{N}_{\dot{\gamma}}^F \mathcal{N}^{\dot{\gamma}G} + h.c. ; F, G = 1, 2, 3 \\ M_{FG} &= M_{GF} : \text{complex arbitrary otherwise} ; |M| \gg |\mu| \end{aligned} \quad (142)$$

It is the primary breakdown along the direction of  $J_5$  which contrary to all 'mirror complexes' brings on the level of (pseudo-) scalar fields to the foreground the complex bosonic 126 and  $\overline{126}$  representations of SO10

$$\begin{aligned} \mathcal{H}_M &\leftarrow \\ \left( \Phi^{\overline{126}FG} \right)^{\bar{\xi}} &(f_{a16F})_{\dot{\gamma}} (f_{b16G})^{\dot{\gamma}} C \left( \begin{array}{c|cc} 126 & 16 & 16 \\ \xi & a & b \end{array} \right) + h.c. \\ \left( \Phi^{\overline{126}FG} \right)^{\bar{\xi}} &: \text{(p-) scalar fields in the } \overline{126} \text{ representation of SO (10)} \end{aligned} \quad (143)$$

In eq. 143  $C \left( \begin{array}{c|cc} 126 & 16 & 16 \\ \xi & a & b \end{array} \right)$  denotes the coupling coefficients, projecting the symmetric product of two 16-representations of spin (10) to the 126 representation of SO (10) .

The 126 *complex* representation of SO (10) is singled out by the value of  $J_5$  of  $10 = 2 \times 5 \mathcal{N} \mathcal{N}$  .

The relatively complex conjugate representations  $126 \oplus \overline{126}$  are contained in the *real* , *reducible* fivefold antisymmetric tensor representation of SO (10) decomposing into the irreducible pair upon the duality conditions

$$\begin{aligned}
& t [ A_1 A_2 \cdots A_5 ] ; A_1 \dots A_5 = 1, 2, \dots, 10 \\
& t [ A_{\pi_1} A_{\pi_2} \cdots A_{\pi_5} ] = \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \cdots & 5 \\ \pi_1 & \pi_2 & \cdots & \pi_5 \end{array} \right) t [ A_1 A_2 \cdots A_5 ] \\
& \frac{1}{5!} \varepsilon_{A_1 \dots A_5 B_1 \dots B_5} t_{\pm}^{[B_1 B_2 \cdots B_5]} = (\pm i) t_{\pm}^{[A_1 A_2 \cdots A_5]} \\
& \varepsilon_{A_1 \dots A_5 A_6 \dots A_{10}} = \\
& = \text{sgn} \left( \begin{array}{cccc} 1 & 2 & \cdots & 10 \\ \pi_1 & \pi_2 & \cdots & \pi_{10} \end{array} \right) \varepsilon_{A_{\pi_1} \dots A_{\pi_5} A_{\pi_6} \dots A_{\pi_{10}}} \\
& \varepsilon_{1 2 \dots 10} = 1
\end{aligned} \tag{144}$$

Within the complex spin  $(2\nu = 4\tau + 2)$  ,  $\tau = 2, 3, \dots$  series –  $\tau = 2 \leftrightarrow$  spin (10) – the relatively complex conjugate spinorial pair of representations with dimension  $4^\tau \leftarrow 16(64, \dots)$  and the complex selfdual-antiselfdual pair of representations with dimension  $\frac{1}{2} \left( \begin{array}{c} 4\tau + 2 \\ 2\tau + 1 \end{array} \right) \leftarrow 126$  (11.12.13 = 1716,  $\dots$ ) are intrinsically related for  $\tau = 2, 3, 4, \dots$  .

### Some conclusions and questions from section 2-1 .

Q1) Is it enough to consider the primary breakdown and its characteristic, the 'tilt to the left' concerning 3 families, as due *essentially* to spin (10) , which is the *lowest* simple spin group along the complex *orthogonal* chain ?

It has been argued interestingly by Feza Gursey and collaborators [s15-1976], that it is the chain of exceptional groups which encode intrinsically the number 3, which in turn underlies the 3 as the number of (left-chiral) families as well as the strong interaction gauge group  $SU3_c$  .

A1) I think the answer is to the affirmative, since all higher gauge groups , including the exceptional chain and especially E8 , but also spin (14) ,

(18) do *not* explain the #3 of families , rather generate together with even the apparently correct 3 families – for E8 – also mirror families – 3 for E8 , and powers of 2 for the orthogonal chain with  $\tau \geq 3$  .

The tentative conclusion remains, that the structure of families has to be explained outside

spin (10) and *also* outside larger unifying gauge groups containing spin (10) , whereas the origin of neutrino mass is layed out by the lowest member of the complex orthogonal chain  $\rightarrow$  spin (10) .

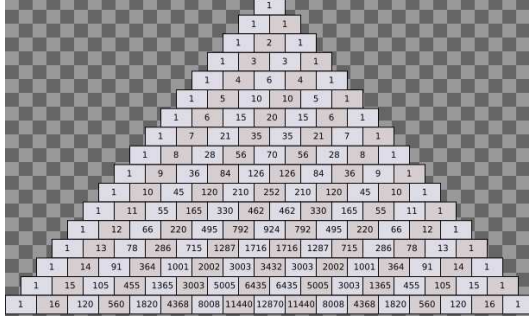
- C4) The two apparently different phenomena of a) 'tilt to the left' and b) baryon number violation are intrinsically associated with the *unusual* sequence of (pseudo)scalar fields generating primary breakdown . We use the notation ( eq. 135 )

$$\begin{aligned}
& \text{spin (10)} \rightarrow \text{SU5} \times \text{U1}_{J_5} \rightarrow \text{SU3}_c \times \text{SU2}_L \times \text{U1}_Y = \text{G}_{s.m.} \\
[16] &= \{1\}_{+5} + \{10\}_{+1} + \{\bar{5}\}_{-3} \\
[\bar{16}] &= \{1\}_{-5} + \{\bar{10}\}_{-1} + \{5\}_{+3} \\
\{ \bar{5} \}_{-3} &= \left[ \begin{pmatrix} \bar{3} \\ 1 \end{pmatrix}_{+\frac{1}{3}} \left[ \begin{pmatrix} - \\ -3 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 2 \end{pmatrix}_{-\frac{1}{2}} \left[ \begin{pmatrix} - \\ -3 \end{pmatrix} \right] \right]
\end{aligned} \tag{145}$$

(p)scalarr SO (10) reprsnt.	active components	induced (a)symmetries	preserved gauge group
$\left. \begin{matrix} [126]_{\mathbb{C}} \\ [\bar{126}]_{\bar{\mathbb{C}}} \end{matrix} \right\} \rightarrow$	$\left. \begin{matrix} \{1\}_{+10} \\ \{\bar{1}\}_{-10} \end{matrix} \right\}$	$P : \text{'tilt to the left'}$ $\mathcal{NN} - \text{mass}, B - L$ $CP \downarrow$	SU5
$[45]_{\mathbb{R}} \} \nearrow \rightarrow$	$\left. \begin{matrix} \{24\}_0 \downarrow \\ \downarrow (1,1)_0 \downarrow \end{matrix} \right\}$	$B, L, \uparrow CP$	$\text{G}_{s.m.}$
$[10]_{\mathbb{R}} \} \nearrow \rightarrow \left[ \begin{matrix} (1,2)_{-\frac{1}{2}} \\ (1,\bar{2})_{+\frac{1}{2}} \end{matrix} \right] \left[ \begin{matrix} -3 \\ +3 \end{matrix} \right]$		$\mathcal{N}\nu \text{ mass } \uparrow$ $\bar{q}q \text{ mass } CKM$ $CP \uparrow$	$\text{SU3}_c \times \text{U1}_{e.m.}$

(146)

Pascals triangle  $\binom{n}{k}$  for  $n = 1, 2, \dots, 16$



**Fig 3 : Pascal's triangle**

**1-1a There does not exist a symmetry – within the standard model including gravity and containing only chiral spin  $\frac{1}{2}$  16 families of SO (10) – which could enforce the vanishing of neutrino mass(es) .**

The divergence of the current associated to the global charge B - L for three standard model families of 15 base fields – in the left chiral basis removing – to infinite mass – the 16-th components  $(\mathcal{N})$  pertaining to one full 16-representation of SO (10) [ spin (10) ]

$$\left( \begin{array}{cccc|cccc} u^1 & u^2 & u^3 & \nu & \mathcal{N} & \hat{u}^3 & \hat{u}^2 & \hat{u}^1 \\ d^1 & d^2 & d^3 & e^- & e^+ & \hat{d}^3 & \hat{d}^2 & \hat{d}^1 \end{array} \right)^{\hat{\gamma} \rightarrow L} \quad (147)$$

$$= (f)^{\hat{\gamma}}$$

and admitting a gravitational background field is in this minimal neutrino flavor embedding anomalous , i.e. the global symmetry is broken by winding

gravitational fields [s3-2001] .

$$j_{\varrho} ( B - L )|_{3 \times 15} = \sum_{fmlies} \left[ \begin{array}{l} (u^*)^{\alpha \dot{c}} (\sigma_{\mu})_{\alpha \dot{\gamma}} (u)^{\dot{\gamma} c} - \\ - (\hat{u}^*)^{\alpha c} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\hat{u})^{\dot{\gamma} \dot{c}} \\ + (d^*)^{\alpha \dot{c}} (\sigma_{\mu})_{\alpha \dot{\gamma}} (d)^{\dot{\gamma} c} - \\ - (\hat{d}^*)^{\alpha c} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\hat{d})^{\dot{\gamma} \dot{c}} \\ - (e^-)^{* \alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (e^-)^{\dot{\gamma}} + \\ + (e^+)^{* \alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (e^+)^{\dot{\gamma}} \\ - (\nu)^{* \alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\nu)^{\dot{\gamma}} \end{array} \right] e_{\varrho}^{\mu} \quad (148)$$

$g_{\varrho \tau} = e_{\varrho}^{\mu} \eta_{\mu \nu} e_{\tau}^{\nu}$  : metric ;  $e_{\varrho}^{\mu}$  : vierbein ;

$*$  : hermitian operator conjugation ;  $(u^*)^{\alpha \dot{c}} \equiv (u^{\dot{a} c})^*$

$\eta_{\mu \nu} = diag ( 1, -1, -1, -1 )$  : tangent space metric

$c ( \dot{c} )$  : color and anticolor ;  $c = 1, 2, 3$

The contribution of charged fermion (pairs)  $q, \hat{q}$  ;  $e^{\mp}$  can be combined to vector currents – Dirac doubling –  $\bar{q} \gamma_{\mu} q$  ;  $\bar{e} \gamma_{\mu} e$  with  $q \rightarrow u, d, c, s, t, b$  ;  $e \rightarrow e^-, \mu^-, \tau^-$  .

The anomalous Ward identity for the B - L current ( - density ) defined in eq. 148 takes the form

$$\begin{aligned} D^4 x \sqrt{|g|} D^{\varrho} j_{\varrho} ( B - L )|_{3 \times 15} &= 3 \hat{A}_1 ( X ) \\ \hat{A}_1 ( X ) &= -\frac{1}{24} tr X^2 ; (X)^a_b = \frac{1}{2\pi} \frac{1}{2} dx^{\varrho} \wedge dx^{\tau} (R^a_b)_{\varrho \tau} \\ (R^a_b)_{\varrho \tau} &: \begin{cases} \text{Riemann curvature tensor} \\ \text{mixed components : } ^a_b \rightarrow \text{tangent space} \\ \mu \nu \rightarrow \text{covariant space} \end{cases} \\ D^{\varrho} j_{\varrho} ( B - L )|_{3 \times (16)} &= 0 \end{aligned} \quad (149)$$

Before discussing the extension

$$j_{\varrho} ( B - L )|_{3 \times (15)} \rightarrow j_{\varrho} ( B - L )|_{3 \times (16)}$$

which renders the latter current conserved, lets define the quantities appearing in eq. 149 :

$$\begin{aligned} (R^{\overset{a}{b}})_{\varrho \tau} &= e_{\mu}^{\overset{a}{b}} e_{b \nu} (R^{\mu}_{\nu})_{\varrho \tau} ; e_{b \nu} = \eta_{bb'} e_{\nu}^{b'} \\ (R^{\mu}_{\nu})_{\varrho \tau} &= (\partial_{\varrho} \Gamma_{\tau} - \partial_{\tau} \Gamma_{\varrho} + \Gamma_{\varrho} \Gamma_{\tau} - \Gamma_{\tau} \Gamma_{\varrho})^{\mu}_{\nu} \\ (\Gamma^{\mu}_{\nu})_{\tau} &: \text{matrix valued } (GL(4, \mathbb{R})) \text{ connection ; minimal here} \end{aligned} \quad (150)$$



For clarity eq. 149 is repeated below

$$\begin{aligned}
d^4 x \sqrt{|g|} D^\varrho j_\varrho (B - L) |_{3 \times 15} &= 3 \hat{A}_1 (X) \\
\hat{A}_1 (X) &= -\frac{1}{24} \text{tr} X^2; (X)^a_b = \frac{1}{2\pi} \frac{1}{2} dx^\varrho \wedge dx^\tau (R^a_b)_{\varrho\tau} \\
(R^a_b)_{\varrho\tau} &: \begin{cases} \text{Riemann curvature tensor} \\ \text{mixed components : } a_b \rightarrow \text{tangent space} \\ \mu\nu \rightarrow \text{covariant space} \end{cases} \\
D^\varrho j_\varrho (B - L) |_{3 \times (16)} &= 0
\end{aligned} \tag{151}$$

In eq. 149  $\hat{A}(X \rightarrow \lambda) = \frac{1}{2} \lambda / \sinh(\frac{1}{2} \lambda)$  denotes the Atiyah - Hirzebruch character or  $\hat{A}$  - genus [s4-1966] with its integral over a compact, euclidean signatred closed manifold  $M_4$ , capable of carrying on SO4 - spin structure, becomes the index of the associated *elliptic* Dirac equation

$$\int \hat{A}(X_E) = n_R - n_L = \text{integer} \tag{152}$$

In eq. 152  $n_{R,L}$  denote the numbers of right - and left - chiral solutions of the Dirac equation on  $M_4$ . The index  $E \rightarrow X_E$  shall indicate the euclidean transposed curvature 2 - form, and is *adapted* here to physical curved and uncurved space time.

For the latter case the first relation in eq. 149 yields the integrated form – in the limit of infinitely heavy

$\mathcal{N}_F$  (eq. 147) –

$$\Delta_{R-L} n_\nu = \int d^4 x \sqrt{|g|} D^\mu j_\mu^{B-L(15)} = \textcolor{red}{3} \textcolor{blue}{\Delta n(\hat{A})} \tag{153}$$

$$3 = \text{number of families} = \text{odd} \quad ; \quad m_{\nu_F} \rightarrow 0$$

In eq. 153  $\Delta_{R-L} n_\nu$  denotes the difference of right - chiral ( $\hat{\nu}$ )<sup>5</sup> and left - chiral ( $\nu$ ) flavors between times  $t \rightarrow \pm \infty$ .

Here a subtlety arises *precisely* because the number of families on the level of  $G_{SM}$  is odd, and the light neutrino flavors are not 'Dirac - doubled', which according to eq. 153 could potentially lead to a change in fermion number being odd, which violates the rotation by  $2\pi$  symmetry, equivalent to  $\hat{\Theta}^2$  ( $CPT^2$ ), *unless*<sup>6</sup>

$$\Delta n(\hat{A}) = \text{even} \quad (\sqrt{\text{ for } \dim = 4 \bmod 8}) \tag{154}$$

We now turn to the SO (10) inspired cancellation of the gravity induced anomaly, giving rise to the completion of neutrino flavors to 3 families of

<sup>5</sup>  $\hat{\nu}_\alpha \equiv \varepsilon_{\alpha\beta} (\nu^*)^\gamma$ ;  $\varepsilon = i\sigma_2$ ; (2nd Pauli matrix) stands for the left-chiral neutrino fields transformed to the right-chiral basis.

<sup>6</sup> The obviously nontrivial relation between the compact Euclidean - and noncompact asymptotic and locality restricted form of the index theorem involves not clearly formulated *boundary conditions*.

16-plets , sometimes called 'right-handed' neutrino flavors, denoted  $\mathcal{N}$  in the left-chiral basis in eq. 147 [s5-2007]

$$\left( \begin{array}{cccc|cccc} \textcolor{red}{u}^1 & \textcolor{green}{u}^2 & \textcolor{blue}{u}^3 & \nu & \mathcal{N} & \textcolor{blue}{\hat{u}}^3 & \textcolor{green}{\hat{u}}^2 & \textcolor{red}{\hat{u}}^1 \\ \textcolor{red}{d}^1 & \textcolor{green}{d}^2 & \textcolor{blue}{d}^3 & e^- & e^+ & \textcolor{blue}{\hat{d}}^3 & \textcolor{green}{\hat{d}}^2 & \textcolor{red}{\hat{d}}^1 \end{array} \right)^{\dot{\gamma} \rightarrow L} \quad (155)$$

$$= (\textcolor{blue}{f})^{\dot{\gamma}}$$

$$j_{\varrho} (B - L)|_{3 \times 15} \rightarrow j_{\varrho} (B - L)|_{3 \times 16} \quad (156)$$

$$\begin{aligned} d^4 x \sqrt{|g|} D^{\varrho} j_{\varrho} (B - L)|_{3 \times 15} &= 3 \hat{A}_1 (X) \\ \hat{A}_1 (X) &= -\frac{1}{24} \text{tr} X^2; (X)^a_b = \frac{1}{2\pi} \frac{1}{2} dx^{\varrho} \wedge dx^{\tau} (R^a_b)_{\varrho\tau} \\ (R^a_b)_{\varrho\tau} &: \begin{cases} \text{Riemann curvature tensor} \\ \text{mixed components : } ^a_b \rightarrow \text{tangent space} \\ \mu\nu \rightarrow \text{covariant space} \end{cases} \\ D^{\varrho} j_{\varrho} (B - L)|_{3 \times (16)} &= 0 \rightarrow \end{aligned} \quad (157)$$

$$\begin{aligned} j_{\varrho} (B - L)|_{3 \times 15} \rightarrow j_{\varrho} (B - L)|_{3 \times 16} &= \\ \sum_{fmlies} \left[ \begin{array}{l} (u^*)^{\alpha\dot{c}} (\sigma_{\mu})_{\alpha\dot{\gamma}} (u)^{\dot{\gamma}c} - \\ - (\hat{u}^*)^{\alpha\dot{c}} (\sigma_{\mu})_{\alpha\dot{\gamma}} (\hat{u})^{\dot{\gamma}\dot{c}} \\ + (d^*)^{\alpha\dot{c}} (\sigma_{\mu})_{\alpha\dot{\gamma}} (d)^{\dot{\gamma}c} - \\ - (\hat{d}^*)^{\alpha\dot{c}} (\sigma_{\mu})_{\alpha\dot{\gamma}} (\hat{d})^{\dot{\gamma}\dot{c}} \\ - (e^-)^{* \alpha} (\sigma_{\mu})_{\alpha\dot{\gamma}} (e^-)^{\dot{\gamma}} + \\ + (e^+)^{* \alpha} (\sigma_{\mu})_{\alpha\dot{\gamma}} (e^+)^{\dot{\gamma}} \\ - (\nu)^{* \alpha} (\sigma_{\mu})_{\alpha\dot{\gamma}} (\nu)^{\dot{\gamma}} + \\ \underbrace{(\mathcal{N})^{\alpha\dot{c}} + (\sigma_{\mu})_{\alpha\dot{\gamma}} (\mathcal{N})^{\dot{\gamma}c}} \end{array} \right] e^{\mu}_{\varrho} \quad (158) \end{aligned}$$

$g_{\varrho\tau} = e^{\mu}_{\varrho} \eta_{\mu\nu} e^{\nu}_{\tau}$  : metric ;  $e^{\mu}_{\varrho}$  : vierbein ;  
 $*$  : hermitian operator conjugation ;  $(u^*)^{\alpha\dot{c}} \equiv (u^{\dot{a}c})^*$  ;  
 $\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)$  : tangent space metric  
 $^c (^{\dot{c}})$  : color and anticolor ;  $c = 1, 2, 3$

$$D^{\varrho} j_{\varrho} (B - L)|_{3 \times (16)} = 0$$

Let me illustrate the triple doubling in the elimination of the anomaly in the covariant divergence of  $j_{\varrho} (B - L)|_{3 \times 15}$  in eq. 148 as seen through

the left-chiral basis , repeating only the  $\nu$  ,  $\mathcal{N}$  components of the B - L current in eq. 158

$$j_{\varrho} ( B - L )|_{3 \times 16} = \sum_{fmlies} \left[ \begin{array}{c} \dots \\ - (\nu)^*{}^{\alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\nu)^{\dot{\gamma}} + \\ \underbrace{(\mathcal{N})^*{}^{\alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\mathcal{N})^{\dot{\gamma}}} \end{array} \right] \quad (159)$$

	$\nu_F^{\dot{\gamma}}$	$\mathcal{N}_F^{\dot{\gamma}}$
$B - L$	-1	+1

;  $F = 1, 2, 3$  family

**1-2a There does not exist a symmetry – within the standard model including gravity and containing only chiral 16 families of SO (10) – enforcing the vanishing of neutrino mass(es), yet chiral extensions can accomplish this**

Here I briefly describe one such extension. It consists of replacing in each family the SO (10) induced  $\mathcal{N}_F$  flavors by four alternative ( sterile )  $\mathcal{X}_{J=2,3,4,5;F}$  flavors, singlets under the electroweak gauge group with genuinely chiral B - L charges, changing the structure in eq. 159 to

$$j_{\varrho} ( B - L )|_{3 \times 19} = \sum_F \left[ \begin{array}{c} \dots \\ - (\nu)^*{}^{\alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\nu)^{\dot{\gamma}} + \\ + \sum_{J=2}^5 (\chi)_J (\mathcal{X}_J)^*{}^{\alpha} (\sigma_{\mu})_{\alpha \dot{\gamma}} (\mathcal{X}_J)^{\dot{\gamma}} \end{array} \right] \quad (160)$$

	$\nu_F^{\dot{\gamma}} = \mathcal{X}_{1,F}^{\dot{\gamma}}$	$\mathcal{X}_{2,F}^{\dot{\gamma}}$	$\mathcal{X}_{3,F}^{\dot{\gamma}}$	$\mathcal{X}_{4,F}^{\dot{\gamma}}$	$\mathcal{X}_{5,F}^{\dot{\gamma}}$
$B - L$ $= (\chi)_J$	-1	-5	-9	7	8

;

$$F = 1, 2, 3 \text{ family}$$

$$J = 1, 2, \dots, 5$$

The genuinely chiral couplings  $(\chi)_{J=1,\dots,5} = [ -1 , -5 , -9 ; 7 , 8 ]$  for neutrino flavors as shown in eq. 160 with 5 chiral base flavors merit some comments :

- 1) a sequence of charges  $(\chi)_J$  ,  $J = 1, \dots, N$  with respect to the left-chiral basis – to be specific – shall be called *genuinely* chiral , if none

of the charges vanishes and no pairs of opposite charge  $[\pm(\chi)]$  are admitted.

- 2) the absence of an anomaly of the associated chiral current, of the form given for neutrino flavors in eqs. 148, 156 and 160 including also gravitational fields leads *in 4 dimensions* to the two conditions

$$\sum_J^N (\chi)_J = 0, \quad \sum_J^N [(\chi)_J]^3 = 0 \quad (161)$$

- 3) there does not exist a genuinely chiral set  $\{(\chi)_J, J = 1, \dots, N\}$  for  $N < 5$ .

For  $N = 3, 4$  it is equivalent to show that the two equations

$$\begin{aligned} A + B &= C + D, \quad A^3 + B^3 = C^3 + D^3 \quad \rightarrow \\ A &= x - a, \quad B = x + a, \quad C = x - b, \quad D = x + b \rightarrow \\ x a^2 &= x b^2 \quad \rightarrow \quad \{ x = 0 \text{ or } x \neq 0; b = \pm a \} \end{aligned} \quad (162)$$

have no solution, satisfying the conditions for genuine chirality.

- 4) There are infinitely many solutions for  $N \geq 5$ , with chiral charges relatively irrational as well as rational. For integer values and  $N = 5$  with the norm  $|(\chi)| = \sum |(\chi)_J|$  the solution with smallest norm is unique up to an overall change of sign<sup>7</sup>

$$(\chi)_J = [-1, -5, -9; 7, 8] \quad (163)$$

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<sup>7</sup> It is due to Paul Frampton, on a beautiful morning in 1993, along the coastal range above the mediterranean sea near Cassis, France.

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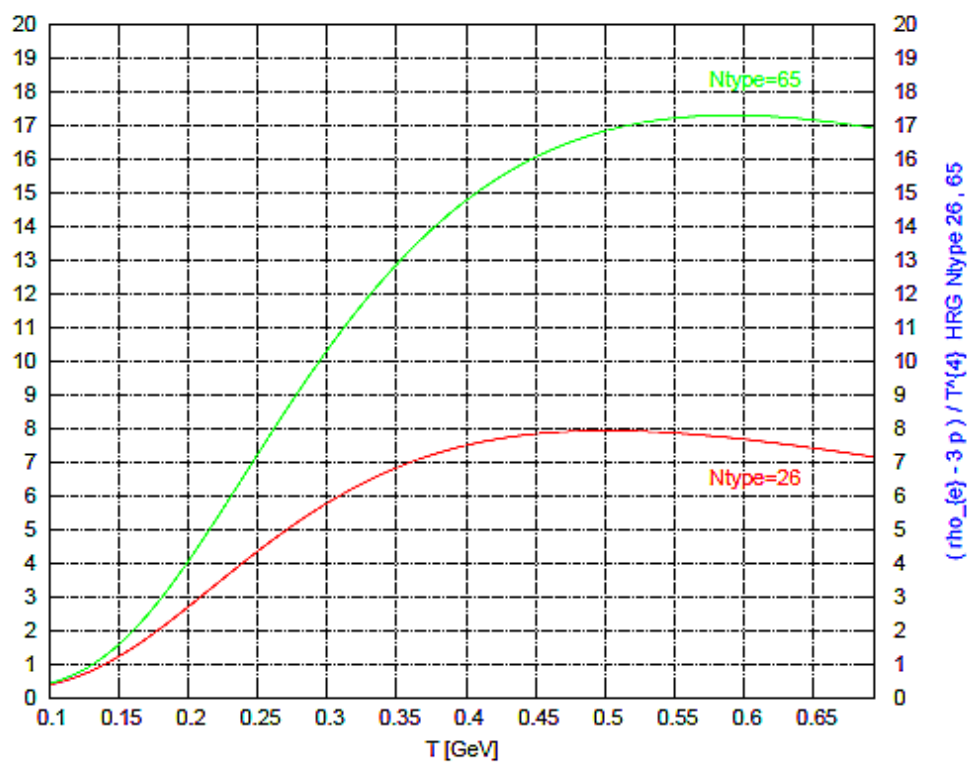


Fig. 3 : The thermal quantity representing the trace anomaly – dscale  $(T) = (\varrho_e - 3p) / T^4$  is shown for  $0.1 \text{ GeV} \leq T \leq 0.7 \text{ GeV}$ . The comparison of HRG collections Ntype=65 and 26 shows the sensitive temperature regions beyond temperatures where chemical freeze out takes place –  $T \sim 150 - 170 \text{ MeV}$  for Pb - Pb collisions at SPS and Au - Au collisions at RHIC.